

Preference Robust Ordinal Priority Approach and its Satisficing Extension for Multi-Attribute Decision-Making with Incomplete Information

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Abstract

Ordinal Priority Approach (OPA) has recently been proposed to deal with multi-attribute decision-making (MADM) for determining the weights of experts, attributes, and alternatives under incomplete preference information. This study presents an equivalent reformulation of OPA related to rank order centroid weights, further providing its closed-form solution and decomposability. Building on these properties, we propose a Preference Robust Ordinal Priority Approach (OPA-PR) utilizing a two-stage optimization framework to generalize the utility structure and counter the ambiguity within the ranking parameters and utility preferences. In the first stage, OPA-PR elicits the worst-case utility functions across all experts and attributes from utility preference ambiguity sets characterized by monotonicity, normalization, concavity, Lipschitz continuity, and moment-type preference elicitation. In the second stage, OPA-PR optimizes decision weights based on the elicited utility functions, considering the worst-case ranking parameters indicated by support functions. We suggest a piecewise linear approximation for the utility preference ambiguity sets to derive the tractable reformulation of OPA-PR, verified by the error bounds for optimal outcomes in both stages. Critical properties of OPA-PR are provided, including closed-form solution, invariance of optimal weight disparity, and risk preference independence. Moreover, considering the potential ranking parameter misspecification, we develop a robust satisficing extension of OPA-PR based on its closed-form solution, OPA-PRS, with a novel decision criterion-fragility measure of weight disparity. The effectiveness of OPA-PR and OPA-PRS is validated through a numerical experiment of the emergency supplier selection problem.

Keywords: Multi-attribute decision-making, Incomplete information, Utility preference ambiguity, Model misspecification, Preference robust ordinal priority approach, Robust satisficing

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1. Introduction

Over the past few decades, multi-attribute decision-making (MADM) has emerged as a critical area of operations research, particularly for tackling complex discrete decision problems characterized by conflicting objectives and diverse data (Colorni, 2024). A classic MADM problem involves a decision-maker (DM) determining the optimal alternative from a set of alternatives or establishing global rankings based on multiple attributes evaluated by various experts (Greco et al., 2024). Given expert set $\mathcal{I} := \{1, \dots, I\}$, attribute set $\mathcal{J} := \{1, \dots, J\}$, and alternative set $\mathcal{K} := \{1, \dots, K\}$, the multi-attribute group evaluation score Z_k for alternative $k \in \mathcal{K}$ can be determined by a function $F : \mathbb{R} \times \mathbb{R}^I \times \mathbb{R}^J \rightarrow \mathbb{R}$ that has an associated collection of the weights for all expert $i \in \mathcal{I}$ and attribute $j \in \mathcal{J}$ and corresponding utility function:

$$Z_k = F(v_{jk}) = \sum_{i \in \mathcal{I}} w_i \sum_{j \in \mathcal{J}} w_j u_{ij}(v_{jk}), \quad \forall k \in \mathcal{K},$$

where v_{jk} denotes the performance of alternative k under attribute j , $u_{ij} : \mathbb{R} \rightarrow \mathbb{R}$ represents the utility function for attribute j of expert i that mapping the alternative's performance score v_{jk} to the utility value of expert i under attribute j , w_i and w_j are the weight for expert i and attribute j , respectively.

Classical research is mainly based on the weights and utility functions obtained through sophisticated heuristic methods with complete information assumption, which means one can always elicit precise weights and utility functions with collected data (Ahn, 2017). If DM can provide all the necessary information to resolve MADM problems, the prior (sophisticated) methods based on precise data are advisable. However, challenges arise when the information is incomplete in specifying the weights and utility functions (Guo et al., 2024). Correspondingly, there are two different sources of uncertainty that can plague a MADM problem arising from incomplete information: parameter uncertainty and preference uncertainty (Lu and Shen, 2021). In parameter uncertainty, the DM is unaware of the exact nature of the model parameters influencing their decision, a common issue in collecting alternative performance. Preference uncertainty occurs when a decision must be made but the DM's preferences regarding trade-offs among different incomparable outcomes are not completely determined, which is typical in eliciting the utility functions for alternative performance and weights for experts and attributes. This preference uncertainty in the utility structure, referred to as utility preference ambiguity in decision science literature, is an endogenous uncertainty that arises from variability, contradiction, and difficulties in accurately defining preferences with incomplete information (Hu et al., 2024). In addition to these uncertainties, it is essential to acknowledge the model misspecifications resulting from the simplified approximations used in decision-making models. Each model is a simplification or approximation intended to clarify or improve the understanding of some specific underlying phenomenon. Overall, the above uncertain situations with incomplete information have general implications in the contexts characterized by time constraints,

inadequate data, and limited domain knowledge and cognitive burden of experts (Zayat et al., 2023).

In this study, based on Ordinal Priority Approach (OPA), a promising MADM methods for incomplete preference information, we propose Preference Robust Ordinal Priority Approach (OPA-PR) and its robust satisficing extension (OPA-PRS) to counter the parameter uncertainty, utility preference ambiguity, and model misspecification. To effectively solve OPA-PR and OPA-PRS for practical usage, we also develop a piecewise linear approximation (PLA) scheme to derive the tractable reformulation with error bound guarantees.

1.1. Related Literature

In this section, we aim to position our contributions in the literature by briefly reviewing relevant studies on MADM with incomplete preference information. To provide contextual structure, we identify two primary streams of MADM with incomplete preference information since 2007: optimization-based methods and extreme point-based methods, as well as OPA, which serves as the baseline for this study.

The optimization-based methods aims to solve the optimal weight assignment under the constraints of incomplete preference information, represented by DEA preference voting model (DEA-PVM) and robust ordinal regression (ROR). DEA-PVM is a typical approach based on social choice theory. It maximizes the total score of each alternative that are defined as the weighted sum of their votes across all ranking positions (Ahn, 2024a). In this process, it considers the voting performance of other alternatives, essentially serving as a relative benchmarking method. The classical model of DEA-PVM is the three preference voting models proposed by Wang et al. (2007): two linear models evaluate alternatives using a common set of weights, while one nonlinear model assesses alternatives based on different weights. Currently, DEA-PVM has been expanded to include various types of preference information and constraints, such as ratio scales (Izadikhah and Farzipoor Saen, 2019), decreasing and convex sequences (Llamazares, 2016), cross-evaluation (Sharafi et al., 2022), and exclusion of inefficient candidates (Baranwal and Vid-yarthi, 2016). ROR is a typical modern method within the artificial intelligence paradigm (Greco et al., 2008). ROR, based on multi-attribute utility theory, identifies and utilizes a complete set of compatible instances, represented by necessary and possible weak preference relations, of value functions that reflect the preferences of all DMs involved. It employs piecewise-linear marginal value functions, characterized by breakpoints when partial preference information is present, such as in pairwise and intensity comparisons. Subsequently, ROR identifies the most robust feasible weight disparities based on the partial preference information. Currently, ROR has been adapted in various ways to integrate with other methods, including stochastic multiobjective acceptability analysis (SMAA) (Corrente et al., 2016), non-additive value functions (represented by the Choquet integral and Sugeno integral (Beliakov et al., 2020)), and outranking relation preference models (such as ELECTRE (Corrente et al., 2017) and PROMETHEE (Kadziński

et al., 2012)).

The extreme point-based methods aim to find the extreme points that characterize the weights incorporating in a set of incomplete preference information (Ahn, 2015). Once identified, the final rankings of alternatives are determined by multiplying the extreme points by the attribute values of the alternatives. Ahn (2015) proposed a straightforward method for finding the extreme points of common types of incomplete preference information in the literature, including weakly ordered relations, ratio scales, absolute differences, and lower bounds on weights. Ahn (2017) transformed the coefficient matrices of incomplete preference information into a class of \mathbf{M} -matrices to identify extreme points, subsequently minimizing the squared deviations from the extreme points to approximate the weights. Furthermore, Ahn (2024b) derived a dual linear programming problem to obtain closed-form solutions, identifying extreme points derived from a set of (strictly) ranked preference information. Additionally, a prevalent weight elicitation approach involves rank-based surrogate weights, where each surrogate weight can be uniquely represented by a set of extreme points (Burk and Nehring, 2023). Notable, there reveals an evolving convergence of DEA preference voting models and extreme point-based methods (Ahn, 2024a). For instance, Llamazares (2024) proposed explicit expressions for weights of various simplex centroids in ranking voting systems inspired by specific simplex centroids of ROC weights.

Overall, the studies discussed above seek to engage constructively and transparently with DMs to accurately elicit and represent their evolving preferences while effectively managing imperfect preference information that may be partial, inconsistent, unstable, or uncertain. However, these studies often overlook DM's risk-averse preferences regarding these uncertainties, which include parameter uncertainty, utility function ambiguity, and potential model misspecifications. It is critical to recognize the prevalence and potential impact of evaluation scale distortions resulting from differing risk preferences on decision outcomes. On the other hand, some studies subjectively assume a piecewise linear utility function based on multi-attribute utility theory, which is a simplified approximation to the true utility function. Theoretically, This subjective assumption fails to provide error bounds and performance guarantees for utility function approximation, going further to provide an effective preference elicitation scheme that reduces these errors.

OPA, proposed by Ataei et al. (2020), is a promising optimization-based method for addressing multi-attribute decision problems with incomplete information. It frames the weight elicitation problem as a linear programming model within a normalized weight space with strong dominance relations (or refers to ordinal preference), allowing for the simultaneous determination of weights for experts, attributes, and alternatives (Wang, 2024a). OPA utilizes ordinal data as model inputs, which are more readily available and stable compared to the cardinal values and pairwise comparisons used in other MADM methods.

Unlike traditional approaches, OPA eliminates the need for data standardization, expert opinion aggregation, and prior weight determination. Recently, several extensions of OPA have emerged, including fuzzy OPA (OPA-F) (Mahmoudi et al., 2022c; Pamucar et al., 2023), rough set OPA (OPA-RS) (Du et al., 2023; Kucuksari et al., 2023), grey OPA (OPA-G) (Sadeghi et al., 2024), and robust OPA (OPA-R) (Mahmoudi et al., 2022a) to address data uncertainty; partial OPA (OPA-P) (Wang et al., 2024) for managing Pareto dominance relations; TOPSIS-OPA (Mahmoudi et al., 2021), DEMATEL-OPA (Zhao et al., 2024), and DGRA-OPA-P (Wang, 2024b) for large-scale group decision-making; and DEA-OPA (Mahmoudi et al., 2022b; Cui et al., 2024) for relative efficiency analysis. As a result, OPA has gained significant attention and is being applied across various domains, such as supplier evaluation, stock portfolio assessment, blockchain technology analysis, and environmental efficiency evaluation (Javed and Du, 2023). However, despite its practical applications, research on the fundamental properties of OPA is limited (Mahmoudi and Javed, 2023a,b). Consequently, the lack of theoretical analysis of structural characteristics of OPA prevents current research from effectively addressing preference ambiguity and the broader utility forms it represents. Existing extensions primarily involve combining OPA with other MADM methods or related theories, such as grey system theory, rough set theory, and fuzzy theory, without the underlying logic of methodology of OPA (Wang, 2024a). Consequently, no studies explore the extension of OPA in the context of utility preference ambiguity and model misspecification. In contrast, our discussion is motivated by the underlying insights of OPA derived from its structural characteristics and fundamental properties, providing a foundation for extending OPA to tackle parameter ambiguity, utility preference ambiguity, and model misspecification.

1.2. Contributions

This study extends OPA to scenarios involving parameter uncertainty, utility preference ambiguity, and model misspecification in MADM under incomplete information. The main contributions are summarized as follows:

- We formally derive the equivalent reformulation of OPA, demonstrating that OPA elicits weights within a normalized weight space based on rank order centroid weights for alternatives, which intuitively reveals the structure characteristic of OPA. Additionally, we provide the closed-form solution of OPA and show its decomposability. This decomposability implies that the optimal weights of OPA can be determined by either using the closed-form solution or solving the linear programming problem for individual experts separately and then aggregating the results based on expert weights.
- Based on the derived equivalent reformulation of OPA, we propose OPA-PRS within a two-stage

optimization framework that addresses decision-making scenarios characterized by parameter uncertainty and utility preference ambiguity. Such uncertainty is mitigated by performing worst-case decision-making under a set of plausible utility functions and ranking parameter supports for all attributes and experts, which accommodates the design of customized preferences. In the first stage, we elicit worst-case utility functions for all experts and attributes from their corresponding ambiguity sets, incorporating properties such as monotonicity, normalization, concavity, Lipschitz continuity, and moment-type preference elicitation (i.e., a general utility preference structure). The second stage then optimizes the decision weights based on the elicited worst-case utility functions, considering the worst-case ranking parameters for all experts and attributes. We derive its closed-form solution and demonstrate properties such as the invariance of optimal weight disparity scalar and risk preference independence. We also show that the OPA-R proposed by [Mahmoudi et al. \(2022a\)](#) is a special outcome of the OPA-PR modeling prescription when the worst-case utility takes the rank order centroid weights and attribute rankings are based on discrete uncertain scenarios.

- We derive the tractable reformulation of OPA-PR with infinite dimensions. Specifically, motivated by multi-attribute utility theory, PLA is employed to approximate the utility preference ambiguity sets, especially the moment-type preference elicitation, to derive the worst-case utility function in the first-stage problem. We demonstrate that PLA introduces no additional errors when the ambiguity sets of moment-type preference is elicited through the step-like preference information, such as deterministic utility comparison and stochastic lottery comparison. We also derive the error bounds for optimal weights in the second-stage problem deriving from the piecewise linear approximated worst-case utility functions, which provides the theoretical foundation for PLA and the corresponding preference elicitation strategy design. The parameter uncertainty in attribute rankings is addressed using a robust optimization approach.
- We also introduce OPA-PRS to counter potential misspecifications of expert ranking parameters. The key idea behind OPA-PRS is to allow for a trade-off: by tolerating some loss in weight disparity performance, we can enhance robustness against the misspecification of expert ranking. To this end, we propose the fragility measure for weight disparity, which is a novel decision criterion within the weight elicitation research domain. We demonstrate that this criterion is functional with good properties, including upper semi-continuity, monotonicity, positive homogeneity, superadditivity, and pro-robustness. These characteristics not only underscore the practical implications of the fragility measure for weight disparity but also ensure the tractability of the OPA-PRS formulation.

1.3. Organization

The structure of this paper is organized as follows: Section 2 outlines the preliminaries of OPA. Section 3 proposes the unified framework and tractable reformulation of OPA-PR. Section 4 introduces OPA-PRS and its corresponding decision criterion. Section 5 validates the proposed approach through a case study on emergency supplier selection during the 7.21 mega-rainstorm disaster in Zhengzhou, China. Section 6 gives the conclusions and future directions.

2. Preliminaries

Consider a classical MADM problem where DM needs to select the optimal alternative from K alternatives, $\mathcal{K} := \{1, \dots, K\}$, based on J attributes, $\mathcal{J} := \{1, \dots, J\}$, as evaluated by I experts, $\mathcal{I} := \{1, \dots, I\}$. In OPA, DM initially assigns importance ranking $t_i \in [I]$ to each expert $i \in \mathcal{I}$. Each expert $i \in \mathcal{I}$ then provides the ranking $s_{ij} \in [J]$ for each attribute $j \in \mathcal{J}$ and the ranking r_{ijk} for each alternative $k \in \mathcal{K}$ under attribute $j \in \mathcal{J}$. Expert evaluations are conducted independently without group discussions to ensure rankings reflect their personal preferences. By convention, the most important attribute is ranked as 1, the next as 2, and so forth. Notably, OPA supports consistent rankings, and we refer readers to Wang (2024a) for the formulation with consistent rankings. To streamline subsequent discussions, we first define the following three sets:

$$\begin{aligned}\mathcal{X}^1 &:= \{(i, j, k, l) \in \mathcal{I} \times \mathcal{J} \times \mathcal{K} \times \mathcal{K} : r_{ijl} = r_{ijk} + 1, r_{ijk} \in [K - 1]\}, \\ \mathcal{X}^2 &:= \{(i, j, k) \in \mathcal{I} \times \mathcal{J} \times \mathcal{K} : r_{ijk} = K\}, \\ \mathcal{Y} &:= \{(i, j, k) \in \mathcal{I} \times \mathcal{J} \times \mathcal{K}\}.\end{aligned}$$

Intuitively, \mathcal{Y} defines the indexes of all experts, attributes, and alternatives in MADM, while \mathcal{X}_1 represents the set of indexes of alternatives with consecutive rankings under each expert and attribute, and \mathcal{X}_2 denotes the set of indexes of alternatives ranked last under each expert and attribute. Let $A_{ijk}^{(r_{ijk})}$ denote the alternative k ranked r_{ijk} for attribute j by expert i , equipped with the weight w_{ijk} . Then, under expert i and attribute j , DM prefers alternative k over alternative l , and therefore, the weight of alternative k is greater than that of alternative l , i.e.,

$$A_{ijl}^{(r_{ijl})} \succ A_{ijk}^{(r_{ijk})} \Leftrightarrow w_{ijk} > w_{ijl} \Leftrightarrow w_{ijk} - w_{ijl} > 0, \quad \forall (i, j, k, l) \in \mathcal{X}^1. \quad (1)$$

Multiplying both sides of the last inequality in Equation (1) by the ranking parameters to incorporate the effect of expert preference on weight disparity of consecutively ranked alternatives, we have

$$\delta(\mathbf{w}) = \begin{cases} \delta^1(w_{ijk}, w_{ijl}) = t_i s_{ij} r_{ijk} (w_{ijk} - w_{ijl}) > 0, & \forall (i, j, k, l) \in \mathcal{X}^1, \\ \delta^2(w_{ijk}) = t_i s_{ij} r_{ijk} (w_{ijk}) > 0, & \forall (i, j, k) \in \mathcal{X}^2. \end{cases} \quad (2)$$

OPA searches for the maximum weight disparities while reflecting the preferences of experts within the normalized weight space, in terms of the following optimization model:

$$\begin{aligned}
& \max_{\mathbf{w}, z} z, \\
& \text{s.t. } z \leq \delta^1(w_{ijk}, w_{ijl}), \quad \forall (i, j, k, l) \in \mathcal{X}^1, \\
& \quad z \leq \delta^2(w_{ijk}), \quad \forall (i, j, k) \in \mathcal{X}^2,
\end{aligned} \tag{3}$$

where \mathcal{W} denotes the normalized weight space:

$$\mathcal{W} := \left\{ w_{ijk} \in \mathbb{R}_+^{I \times J \times K} : \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K w_{ijk} = 1, w_{ijk} \geq 0, \forall (i, j, k) \in \mathcal{Y} \right\},$$

which is a closed compact set.

Proposition 1 (Ataei et al. (2020)). *Given the rankings of experts t_i for all $i \in \mathcal{I}$, attributes provided by experts s_{ij} for all $(i, j) \in \mathcal{I} \times \mathcal{J}$, and alternatives under attributes provided by experts r_{ijk} for all $(i, j, k) \in \mathcal{Y}$, OPA is formulated as Equation (4).*

$$\begin{aligned}
& \max_{\mathbf{w}, z} z \\
& \text{s.t. } z \leq t_i s_{ij} r_{ijk} (w_{ijk} - w_{ijl}) \quad \forall (i, j, k, l) \in \mathcal{X}^1 \\
& \quad z \leq t_i s_{ij} r_{ijk} (w_{ijk}) \quad \forall (i, j, k) \in \mathcal{X}^2 \\
& \quad \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K w_{ijk} = 1 \\
& \quad w_{ijk} \geq 0 \quad \forall (i, j, k) \in \mathcal{Y}
\end{aligned} \tag{4}$$

Let z^* denote the optimal weight disparity scalar and w_{ijk}^* denote the optimal weight of alternative k under attribute j by expert i . The weights of experts, attributes, and alternatives, denoted as $W^{\mathcal{I}}$, $W^{\mathcal{J}}$, and $W^{\mathcal{K}}$, are then given by Equation (5).

$$\begin{aligned}
W_i^{\mathcal{I}} &= \sum_{j=1}^J \sum_{k=1}^K w_{ijk}^* \quad \forall i \in \mathcal{I} \\
W_j^{\mathcal{J}} &= \sum_{i=1}^I \sum_{k=1}^K w_{ijk}^* \quad \forall j \in \mathcal{J} \\
W_k^{\mathcal{K}} &= \sum_{i=1}^I \sum_{j=1}^J w_{ijk}^* \quad \forall k \in \mathcal{K}
\end{aligned} \tag{5}$$

In the following, we look into the closed-form solution of OPA and its properties. Without loss of generality, we map the alternative index k to the ranking index r corresponding to their ranking position r_{ijk} with $R = K$ and define $\mathcal{H} := \{(i, j, r) \in \mathcal{I} \times \mathcal{J} \times \mathcal{R}\}$.

The following two lemmas provide the foundation for deriving the closed-form solution of OPA.

Lemma 1. *All constraints in Equation (4) are active when it achieves optimality.*

By Lemma 1, the cumulative sum of the last r inequality constraints in ascending order does not influence the optimal solution, which yields:

$$w_{ijr} \geq \frac{1}{t_i s_{ij}} \left(\sum_{h=r}^R \frac{1}{h} \right) z, \quad \forall (i, j, r) \in \mathcal{H}.$$

Thus, OPA admits the equivalent reformulation as shown in Equation (6).

$$\begin{aligned} & \max_{\mathbf{w}, z} z \\ & \text{s.t.} \quad \left(\sum_{h=r}^R \frac{1}{h} \right) z \leq t_i s_{ij} w_{ijr} \quad \forall (i, j, r) \in \mathcal{H} \\ & \quad \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R w_{ijr} = 1 \\ & \quad w_{ijr} \geq 0 \quad \forall (i, j, r) \in \mathcal{H} \end{aligned} \tag{6}$$

We observe that the left-side parameter in the reformulated OPA can be expressed as the product of rank order centroid weights for the alternative ranked r and the number of alternatives, i.e., $Rv_r^{ROC} = \sum_{h=r}^R \frac{1}{h}$ for $r = 1, \dots, R$. In this sense, OPA can be regarded as eliciting weights based on a specific utility structure for alternatives (i.e., rank order centroid weights) within a normalized weight space, aiming to maximize weight disparities while aligning with expert preferences.

Lemma 2. *The reformulated OPA in Equation (6) is equivalent to*

$$\min_{\mathbf{w} \in \mathcal{W}, z} \left\{ z : \left(\sum_{h=r}^R \frac{1}{h} \right) z \geq t_i s_{ij} w_{ijr}, \forall (i, j, r) \in \mathcal{H} \right\}. \tag{7}$$

The following theorem presents the closed-form (analytical) solution of OPA, which enables the computation of optimal weights from the given ranking parameters without solving the LP problem.

Theorem 1. *The closed-form solution of OPA is*

$$z^* = 1 / \left(R \left(\sum_{p=1}^I \frac{1}{p} \right) \left(\sum_{q=1}^J \frac{1}{q} \right) \right), \tag{8}$$

and

$$w_{ijr}^* = \left(\sum_{h=r}^R \frac{1}{h} \right) / \left(t_i s_{ij} R \left(\sum_{p=1}^I \frac{1}{p} \right) \left(\sum_{q=1}^J \frac{1}{q} \right) \right), \quad \forall (i, j, r) \in \mathcal{H}. \tag{9}$$

The following corollary shows that the optimal weights of OPA can be decomposed into components independently influenced by the ranking indices of experts, attributes, and alternatives.

Corollary 1. Let v_l^{ROC} and v_l^{RR} denote the rank order centroid weight and rank reciprocal weight of the object ranked l , given by $v_l^{ROC} = \left(\sum_{h=l}^L \frac{1}{h} \right) / L$ and $v_l^{RR} = 1 / \left(l \sum_{i=1}^L \frac{1}{i} \right)$ for $l = 1, \dots, L$. The optimal weights of OPA can be expressed as $w_{tsr}^* = v_t^{RR} v_s^{RR} v_r^{ROC}$ for all $t \in [I]$, $s \in [J]$, and $r \in [K]$, where t , s , and r are the ranking indices of experts, attributes, and alternatives.

By Corollary 1, we observe that the rank order centroid weights and rank reciprocal weights, common rank-based surrogate weights, are special cases of OPA for determining the weights of alternatives and attributes, respectively. It can also be inferred from Corollary 1 that the optimal weights of OPA can be determined by solving the linear programming problem for individual experts separately and then aggregating them according to expert weights. In this sense, OPA avoids the problem of divergent group consensus often seen in MADM methods like AHP and BWM (Wang, 2024a).

3. Robust Ordinal Priority Approach

This section introduces the preference robust ordinal priority approach (OPA-PR) based on a two-stage optimization framework to counter the ambiguity within utility structures and ranking parameters.

3.1. Unified Framework

Recall that the inequality constraints in Equation (6) involve two types of parameters: the utility structures for alternatives and the ranking parameters for experts and attributes. We can rewrite OPA as:

$$\max_{\mathbf{w} \in \mathcal{W}, z} \{z : \mathbf{f}(z, \mathbf{u}) \preceq \mathbf{g}(\mathbf{w})\}, \quad (10)$$

where $f_{ijr}(z, u_{ijr}) = Ru_{ijr}^{ROC} z$ and $g_{ijr}(w_{ijr}) = t_i s_{ij} w_{ijr}$ for all $(i, j, r) \in \mathcal{H}$. Correspondingly, we motivate the subsequent discussion on the robust extension of OPA under the uncertainty of the utility structures in $\mathbf{f}(z, \mathbf{u})$ (i.e., utility preference ambiguity) and the ranking parameters in $\mathbf{g}(\mathbf{w})$ (i.e., ranking parameter ambiguity).

For utility structure, it is desired to first extend the rank centroid weights for alternatives to a more general utility function that accounts for ambiguity in the preference information of DMs. This can significantly expand the flexibility of the proposed approach by customizing it to reflect different preferences, rather than restricting it to a specific alternative utility structure as in OPA. Regarding the utility preference ambiguity, our focus here is the situation where DMs do not have complete information to uniquely specify the utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ that maps alternative rankings to utility values. However, partial information can be gathered to construct the ambiguity sets of utility functions, denoted as \mathcal{U} , such that the true utility function reflecting the DM's preference falls within the ambiguity set with high likelihood. Regarding ambiguity in ranking parameters, we formulate in a common way of robust optimization (Jin

et al., 2021), i.e., $\mathbf{g}(\mathbf{w}, \zeta)$ for all $\zeta \in \mathcal{V}$, where the ranking parameters t_i and s_{ij} for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$ are related to the scenarios in ambiguity set \mathcal{V}_{ij} indexed by ζ_{ij} . By Corollary 1, we can solve the optimal weights \mathbf{w}_i for each expert $i \in \mathcal{I}$ separately and then aggregating them by the weights of experts to obtain the group optimal weights $\mathbf{w}_i^* = [w_i^T \mathbf{w}_i]$ for expert i . Thus, our focus here is the ambiguity of attribute rankings given by experts, i.e., $\tilde{s}_{ij} \in \mathcal{S}_{ij} \subset \mathcal{V}_{ij}$ for all $(i, j) \in \mathcal{I} \times \mathcal{J}$, without considering the ambiguity of expert ranking.

In OPA-PR, we mitigate the above two types of ambiguity by constructing a set of plausible utility functions and ranking parameter supports and then determining the optimal weights based on the worst-case scenario. Specifically, we formulate OPA-PR as a two-stage optimization problem under uncertainty. The first stage involves eliciting the worst-case utility function u_{ij}^* for all $(i, j) \in \mathcal{I} \times \mathcal{J}$ from a set of plausible utility functions \mathcal{U}_{ij} over the expectation of random return $h(\mathbf{x}, \boldsymbol{\xi})$ of lottery $\mathbf{x} \in \mathbb{R}^m$ indexed by the finite scenario $\boldsymbol{\xi} \in \mathbb{R}^n$ with associated probabilities $p_e = \mathbb{P}[\boldsymbol{\xi} = \boldsymbol{\xi}_e]$ for $e = 1, \dots, E$. In the second stage, we optimize for the optimal decision weights under the worst-case ranking parameters for all $\zeta \in \mathcal{V}$. Thus, we have the following unified framework for OPA-PR:

$$\max_{\mathbf{w} \in \mathcal{W}, z} \left\{ z : \max_{\zeta \in \mathcal{V}} \{ \mathbf{f}(z, \mathbf{u}^*) - \mathbf{g}(\mathbf{w}, \zeta) \} \preceq 0, u_{ij}^* = \arg \min_{u_{ij} \in \mathcal{U}_{ij}} \mathbb{E}_{\mathbb{P}}[u_{ij}(h(\mathbf{x}, \boldsymbol{\xi}))], \forall (i, j) \in \mathcal{I} \times \mathcal{J} \right\}, \quad (11)$$

or equivalently

$$\begin{aligned} & \max_{\mathbf{w}, z} z, \\ & \text{s.t. } Ru_{ij}^*(r)z \leq \min_{\tilde{s}_{ij} \in \mathcal{S}_{ij}} t_i \tilde{s}_{ij} w_{ijr}, \quad \forall (i, j, r) \in \mathcal{H}, \\ & u_{ij}^* = \arg \min_{u_{ij} \in \mathcal{U}_{ij}} \sum_{e=1}^E p_e u_{ij}(h(\mathbf{x}, \boldsymbol{\xi}_e)), \quad \forall (i, j) \in \mathcal{I} \times \mathcal{J}, \\ & \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R w_{ijr} = 1, \\ & w_{ijr} \geq 0, \quad \forall (i, j, r) \in \mathcal{H}. \end{aligned} \quad (12)$$

The OPA-PR decision pipeline is shown in Figure 1.

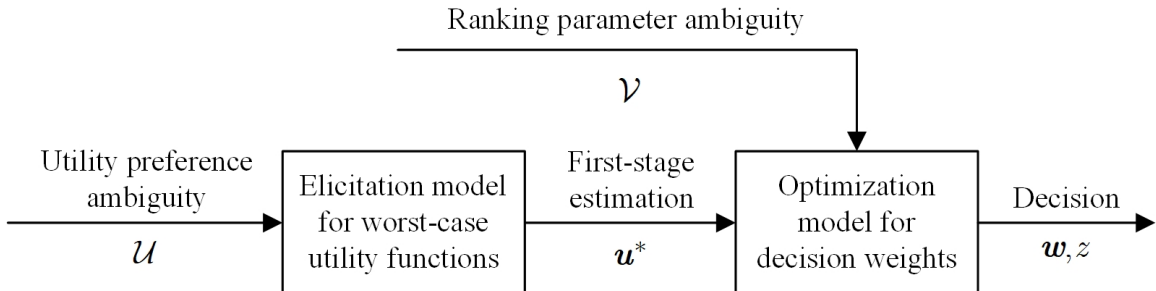


Figure 1: OPA-PR decision pipeline

The key to our success is designing ambiguity sets based on partial preference information of experts and further determining the tractable reformulation of Equation (12).

Remark 1. Although it might be tempting to straightforwardly adopt the joint worst-case scenario of $\zeta \in \mathcal{V}$ and $u \in \mathcal{U}$, i.e., $\max_{\mathbf{w} \in \mathcal{W}, z} \{z : \max_{\zeta \in \mathcal{S}, u \in \mathcal{U}} \{\mathbf{f}(z, \mathbf{u}) - \mathbf{g}(\mathbf{w}, \zeta)\} \preceq 0\}$. However, this formulation optimizes the weight disparity scalar z under the worst-case scenario, which does not adequately capture the behavioral characteristic of DM to confront uncertainty. In contrast, we elucidates the optimal weights when DM demonstrates a conservative (risk-averse) behavior in the face of uncertainty. We optimize the worst-case utility function from the ambiguity set reflecting DM’s preference attitude in the first stage. Although the joint worst-case formulation may be better at optimizing weight disparity scalar z , Equation (11) more interpretable of DM’s behavior than merely pursuing optimization performance.

3.2. Ambiguity Set Design

We begin with designing the ambiguity set for the utility functions related to the first-stage problem of OPA-PR, which is formed by the intersection of the following properties of utility function: monotonicity, normalization, concavity, Lipschitz continuity, and moment-type preference elicitation.

Consider a measurable space (Ω, \mathcal{F}) , where Ω is a finite non-empty set and \mathcal{F} is a σ -algebra on Ω . The finite sample assumption that $\Omega := \{\xi_1, \dots, \xi_E\}$ with $1 \leq E < \infty$ is common in literature related to risk preference, which can be regarded as a discrete sample approximation of the continuous sample space (Wu et al., 2022). We suppose that the input x is bounded on $\Theta := [0, \theta]$. In OPA, we have the ordered ranking sequence τ_r on Θ with $\theta = R$ indexed by $r = 1, \dots, R$. Let \mathcal{U} represent a class of real-valued increasing utility functions, where any $u \in \mathcal{U}$ is piecewise continuously differentiable with a finite number of non-differentiable points at some of the rankings.

Assumption 1 (Monotonicity). $u \in \mathcal{U}$ is monotonic if $x \preceq y$ implies $u(x) \leq u(y)$ for all $x, y \in \Theta$.

Assumption 2 (Normalization). $u \in \mathcal{U}$ is normalized if $u(0) = 0$ and $u(\theta) = 1$.

The monotonicity and normalization of utility functions are common in literature related to decision theory (Wu et al., 2023). The sets of utility functions satisfying monotonicity and normalization are denoted as $\mathcal{U}_{\text{mon}} := \{u \in \mathcal{U} : u \text{ is monotonic}\}$ and $\mathcal{U}_{\text{nor}} := \{u \in \mathcal{U} : u \text{ is normalized}\}$, respectively.

Assumption 3 (Concavity). $u \in \mathcal{U}$ is concave if $u(\lambda x + (1 - \lambda)y) \geq \lambda u(x) + (1 - \lambda)u(y)$ for all $x, y \in \Theta$ and $\lambda \in [0, 1]$.

The concavity of the utility function reflects the DM’s risk aversion, implying that for any random lottery X , DM prefers the certain outcome $\mathbb{E}[X]$ over the lottery X itself (Armbruster and Delage, 2015).

Although this concavity assumption may not apply to all scenarios, it is generally suitable for most decision-making situations (Timonin, 2013). The set of concave utility functions is $\mathcal{U}_{\text{conc}} := \{u \in \mathcal{U} : u \text{ is concave}\}$.

Assumption 4 (Lipschitz continuity). $u \in \mathcal{U}$ is Lipschitz continuous with modulus being bounded by G if $|\nabla u(x) - \nabla u(y)| \leq G \|x - y\|_1$ for all $x, y \in \Theta$.

Lipschitz continuity can be interpreted as the restriction that a finite input cannot lead to an infinite improvement, meaning DM's preferences cannot change too rapidly around any specific input (Guo et al., 2024). It establishes an upper bound G for the the first-order derivative of utility function over Θ . Thus, when the expert specifies their nominal utility function, we can derive G from its first-order derivative, such as the maximum of $\nabla u(\tau) = 1/(\theta)$ for risk-neutral utility function and $\nabla u(\tau) = (\gamma e^{-\gamma\tau}) / (1 - e^{-\gamma})$ for constant absolute risk-aversion utility function over Θ . Additionally, Lipschitz continuity aids in convergence analysis and in establishing error bounds for ambiguity set approximations, which will be shown in Section 3.4. The set of utility functions satisfying Lipschitz continuity is $\mathcal{U}_{\text{lip}} := \{u \in \mathcal{U} : u \text{ is } G\text{-Lipschitz continuous}\}$.

Definition 1. The set of utility functions satisfying moment-type preference elicitation is defined as:

$$\mathcal{U}_{\text{mpre}} := \left\{ u \in \mathcal{U} : -\infty < \int_0^\theta \psi_l(\tau) du(\tau) \leq c_l \text{ for } l = 1, \dots, L \right\}, \quad (13)$$

where $\psi_l : \Theta \rightarrow \mathbb{R}$ are Lebesgue integrable functions and c_l are some given constants for $l = 1, \dots, L$.

The ψ_l for $l = 1, 2, \dots, L$ in Definition 1 reflects partial preference information, which varies across attributes among different experts. It is easy to verify that $\mathcal{U}_{\text{mpre}}$ is a convex set. The moment-type preference elicitation set encompasses various forms of partial preference information discussed in the literature related to decision theory, with several sensible examples presented below.

Example 1 (Deterministic Utility Comparison (Wang, 2024a)). The deterministic utility comparison primarily involves partial preference information for a deterministic utility prospect, including ratio scales, absolute differences, and lower bounds, with the corresponding set of utility function defined as:

$$\begin{aligned} \mathcal{U}_{\text{rs}} &:= \left\{ u \in \mathcal{U} : \int_0^r du(\tau) - \alpha_{l_1} \int_0^{r-1} du(\tau) = 0 \text{ for } l_1 = 1, \dots, L_1 \right\}, \\ \mathcal{U}_{\text{ad}} &:= \left\{ u \in \mathcal{U} : \int_0^r du(\tau) - \int_0^{r-1} du(\tau) = \beta_{l_2} \text{ for } l_2 = 1, \dots, L_2 \right\}, \\ \mathcal{U}_{\text{lb}} &:= \left\{ u \in \mathcal{U} : \int_0^r du(\tau) = \gamma_{l_3} \text{ for } l_3 = 1, \dots, L_3 \right\}, \end{aligned}$$

where \mathcal{U}_{rs} , \mathcal{U}_{ad} , and \mathcal{U}_{lb} represent the set of utility functions with ratio scales, absolute differences, and lower bounds, respectively. In cases of complete preference information, \mathcal{U}_{rs} is commonly applied in

pairwise-comparison-based MADM methods like AHP, ANP, and BWM, while \mathcal{U}_{ad} is prevalent in quasi-distance-based methods like TOPSIS, VIKOR, and EDAS. We can express the above sets using indicator functions as follows:

$$\begin{aligned}\mathcal{U}_{\text{rs}} &:= \left\{ u \in \mathcal{U} : \int_0^\theta (I_{(0,r]}(\tau) - \alpha_{l_1} I_{(0,r-1]}(\tau)) du(\tau) = 0 \text{ for } l_1 = 1, \dots, L_1 \right\}, \\ \mathcal{U}_{\text{ad}} &:= \left\{ u \in \mathcal{U} : \int_0^\theta (I_{(0,r]}(\tau) - I_{(0,r-1]}(\tau)) du(\tau) = \beta_{l_2} \text{ for } l_2 = 1, \dots, L_2 \right\}, \\ \mathcal{U}_{\text{lb}} &:= \left\{ u \in \mathcal{U} : \int_0^\theta I_{(0,r]}(\tau) du(\tau) = \gamma_{l_3} \text{ for } l_3 = 1, \dots, L_3 \right\},\end{aligned}$$

where $I_r(\tau)$ is indicator functions with domain $[0, r]$ with $\tau \in \Theta$. Without loss of generality, we can unify the above sets into the ambiguity set with deterministic utility comparison

$$\mathcal{U}_{\text{duc}} := \left\{ u \in \mathcal{U} : \int_0^\theta \eta_l(\tau) du(\tau) = c_l \text{ for } l = 1, \dots, L \right\},$$

which is consistent with the form of $\mathcal{U}_{\text{mpre}}$. Notably, η_l is a step function with jumps at given constants c_l for $l = 1, \dots, L$.

Example 2 (Stochastic Lottery Comparison (Armbruster and Delage, 2015)). Given any lottery with cumulative distribution $F(\tau)$ on the domain Θ with the expected utility $u(r_1) \leq \mathbb{E}[F(\tau)] \leq u(r_2)$. By integration by parts, we have:

$$\int_0^\theta u(\tau) dF(\tau) = u(\tau)F(\tau) \Big|_0^\theta - \int_0^\theta F(\tau) du(\tau) \geq \int_0^{r_1} du(\tau) \Rightarrow \int_0^\theta (F(\tau) + I_{(0,r_1]}(\tau)) du(\tau) \leq 1,$$

and

$$\int_0^\theta u(\tau) dF(\tau) = u(\tau)F(\tau) \Big|_0^\theta - \int_0^\theta F(\tau) du(\tau) \leq \int_0^{r_2} du(\tau) \Rightarrow \int_0^\theta (F(\tau) + I_{(0,r_2]}(\tau)) du(\tau) \geq 1,$$

where $I_r(\tau)$ is the indicator function and $u(\tau)F(\tau) \Big|_0^\theta = u(\tau)G(\tau) \Big|_0^\theta = 1$ is from the normalization condition. Then, we have the ambiguity set with stochastic lottery comparisons

$$\mathcal{U}_{\text{slc}} := \left\{ u \in \mathcal{U} : \int_0^\theta (F(\tau) + I_{(0,r_1]}(\tau)) du(\tau) \leq 1, \int_0^\theta (-F(\tau) - I_{(0,r_2]}(\tau)) du(\tau) \leq 1 \right\},$$

which is consistent with the form of $\mathcal{U}_{\text{mpre}}$. If the lottery involves pairwise comparisons with corresponding singleton probability, then ψ_j for $j = 1, \dots, L$ in \mathcal{U}_{slc} is a step function.

Example 3 (Stochastic Dominance Relation (Hu and Mehrotra, 2015)). Given any two lotteries with cumulative distribution $F(\tau)$ and $G(\tau)$ with $\tau \in \Theta$. Suppose that DM prefers $F(\tau)$ to $G(\tau)$. By the expected utility theory, we have $\mathbb{E}[F(\tau)] \geq \mathbb{E}[G(\tau)]$, which can be rewrite by integration by parts as

$$\int_0^\theta u(\tau) dF(\tau) \geq \int_0^\theta u(\tau) dG(\tau) \Rightarrow \int_0^\theta [G(\tau) - F(\tau)] du(\tau) \geq 0.$$

Then, we have the ambiguity set with stochastic dominance relation

$$\mathcal{U}_{\text{sdr}} := \left\{ u \in \mathcal{U} : \int_0^\theta [F(\tau) - G(\tau)] du(\tau) \leq 0 \right\},$$

which is consistent with the form of $\mathcal{U}_{\text{mpre}}$.

Example 4 (Pseudo-Metric of Nominal and True Utility (Guo et al., 2024)). Let \hat{u} denote a nominal utility function derived from empirical data or subjective judgment. The true utility function u with elicited preference satisfies the following condition:

$$\underline{d}_l \leq \int_0^\theta \delta_l(\tau) du(\tau) - \int_0^\theta \delta_l(\tau) d\hat{u}(\tau) \leq \bar{d}_l \text{ for } l = 1, \dots, L,$$

where δ_l for $l = 1, \dots, L$ represents elicited partial preference information, such as deterministic utility comparisons, stochastic lottery comparisons, and dominance relations discussed above. Let $\Delta := \{\delta_1, \dots, \delta_L\}$ and assume $\underline{d}_l = \bar{d}_l = d$. The above condition can be rewritten as $\sup_{\delta \in \Delta} \left| \int_0^\theta \delta(\tau) du(\tau) - \int_0^\theta \delta(\tau) d\hat{u}(\tau) \right| \leq d$, which can be interpreted as a pseudo-metric between \hat{u} and u . Consequently, the ambiguity set with pseudo-metric between the nominal and true utility functions is expressed as:

$$\mathcal{U}_{\text{pd}} := \left\{ u \in \mathcal{U} : \sup_{\delta \in \Delta} \left| \int_0^\theta \delta(\tau) du(\tau) - \int_0^\theta \delta(\tau) d\hat{u}(\tau) \right| \leq d \right\},$$

which can be viewed as a ball of utility functions centered at \hat{u} with radius d in the utility function space \mathcal{U} , consistent with the form of $\mathcal{U}_{\text{mpre}}$. A formal definition of the pseudo-metric will be provided in Section 4 for the analysis of error bounds.

The ambiguity set for utility function can be derived from the intersection of the previously discussed sets:

$$\mathcal{U} := \mathcal{U}_{\text{mon}} \cap \mathcal{U}_{\text{nor}} \cap \mathcal{U}_{\text{conc}} \cap \mathcal{U}_{\text{lip}} \cap \mathcal{U}_{\text{mpre}}, \quad (14)$$

where monotonicity, normalization, concavity, and Lipschitz continuity represent global information, while moment-type preference elicitation serves as local customized information of utility function. Notably, the ambiguity sets of utility functions differs from each attribute given by experts, forming the basis for eliciting customized preferences. Thus, for any expert and attribute $(i, j) \in \mathcal{I} \times \mathcal{J}$, we have the first-stage problem of OPA-PR:

$$\rho_{ij} = \min_{u_{ij} \in \mathcal{U}_{ij}} \sum_{e=1}^E p_e u_{ij}(h(\mathbf{x}, \boldsymbol{\xi}_e)). \quad (15)$$

Regarding the ambiguity sets for ranking parameters, it is tempting to straightforwardly consider interval sets for attribute rankings provided by experts, i.e., $\tilde{s}_{ij} \in [\underline{s}_{ij}, \bar{s}_{ij}] = \mathcal{S}_{ij}$ for all $(i, j) \in \mathcal{I} \times \mathcal{J}$. The upper and lower bounds can be directly provided by experts. This approach aligns with some theory frameworks that address input uncertainty, such as fuzzy theory, grey system theory, and rough set theory

(Kucuksari et al., 2023; Sadeghi et al., 2024). Additionally, in practice, determining the penalty level for common ambiguity sets, such as norm-, polyhedron-, and budget-based sets, is not straightforward for DM. This contrasts with operations management scenarios where penalty levels can be derived from large datasets (Greco et al., 2024).

3.3. Formulation

One of the main tasks in this section is to develop a tractable reformulation of Equation (11). To accomplish this, the initial step is to develop efficient computational schemes for the first-stage optimization problem with infinite dimensions to derive the worst-case utility function. Following this, we derive the tractable reformulation of the second-stage optimization problem with ranking parameter uncertainty to determine the optimal weights.

3.3.1. Ambiguity Set Approximation

This section focuses on the PLA of the ambiguity set of utility functions, particularly $\mathcal{U}_{\text{mpre}}$ in \mathcal{U} , which is nonlinear within the space of utility function \mathcal{U} . PLA is of interest for two main reasons. First, the DM's preferences are elicited at discrete rankings in OPA. Connecting utility values at these points to form a piecewise linear utility function is a straightforward way to obtain an approximated utility function. Second, the piecewise linear assumption is widely used in decision theory research (Wu et al., 2023), as seen in various MADM methods such as UTA, MACBETH, and ROR (Greco et al., 2024).

Definition 2. Let $\mathcal{U}^K \subset \mathcal{U}$ where any $u \in \mathcal{U}^K$ is a piecewise linear function with turning points at rankings on Θ . The set of piecewise linear utility functions satisfying moment-type preference elicitation is defined as

$$\mathcal{U}_{\text{mpre}}^K := \left\{ u^K \in \mathcal{U}^K : -\infty < \int_0^\theta \psi_l(\tau) du^K(\tau) \leq c_l \text{ for } l = 1, \dots, L \right\}. \quad (16)$$

For any $u \in \mathcal{U}_{\text{mpre}}$, the corresponding piecewise linear utility function $u^K \in \mathcal{U}^K$ can be constructed by connecting the function values at the endpoints of each interval $[\tau_{r-1}, \tau_r]$ for $r = 1, \dots, R$, which is given by the following proposition.

Proposition 2. Let $\psi_l(\tau)$ be a step function on Θ with jumps at τ_r for $r = 1, \dots, R$. Then, for any $u \in \mathcal{U}_{\text{mpre}}$, there exists a piecewise linear function $u^K \in \mathcal{U}^K$ with $u^K(\tau_r) = u(\tau_r)$ for $r = 1, \dots, R$, such that $u^K \in \mathcal{U}^K$. Specifically, such u^K can be constructed by

$$u^K(\tau) = u(\tau_{r-1}) + \frac{u(\tau_r) - u(\tau_{r-1})}{\tau_r - \tau_{r-1}}(t - \tau_{r-1}), \quad t \in [\tau_{r-1}, \tau_r] \text{ for } r = 2, \dots, R. \quad (17)$$

By Proposition 2, we can introduce the PLA of the first-stage problem of OPA-PR for any expert and attribute $(i, j) \in \mathcal{I} \times \mathcal{J}$:

$$\rho_{ij}^K = \min_{u_{ij} \in \mathcal{U}_{ij}^K} \sum_{e=1}^E p_e u_{ij}^K(h(\mathbf{x}, \boldsymbol{\xi}_e)), \quad (18)$$

where $\mathcal{U}^K := \mathcal{U}_{\text{mon}} \cap \mathcal{U}_{\text{nor}} \cap \mathcal{U}_{\text{conc}} \cap \mathcal{U}_{\text{lip}} \cap \mathcal{U}_{\text{mpre}}^K$. In Section 3.4, we will analyze the error bounds on the optimal value and solution of both stages of OPA-PR introduced by PLA.

3.3.2. Tractable Reformulation

We begin with deriving the worst-case utility function in the first-stage problem of OPA-PR for all experts and attributes.

The following lemma states that, given a set of points in $\mathbb{R}^n \times \mathbb{R}$, a concave function mapping from \mathbb{R}^n to \mathbb{R} can be expressed as the upper envelope of linear functions, forming a piecewise linear concave function that passes through some of those points.

Lemma 3 (Haskell et al. (2018)). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\langle \cdot \rangle$ denote the Euclidean inner product. The following statement hold.*

(1) *f is concave if and only if*

$$f(\mathbf{x}) = \inf_{i \in \mathcal{I}} g_i(\mathbf{x}), \quad \forall \mathbf{x} \in \text{dom} f,$$

where \mathcal{I} is an index set, potentially infinite, and $g_i(\mathbf{x}) = \langle \mathbf{a}_i, \mathbf{x} \rangle + \mathbf{b}_i$ for all $i \in \mathcal{I}$, representing the support function of f at \mathbf{x} for any $\mathbf{a} \in \partial f(\mathbf{x})$.

(2) *For any finite set $\mathcal{O} \subset \mathbb{R}^n$ and values $\{\mathbf{v}_o\}_{o \in \mathcal{O}} \subset \mathbb{R}$, the function $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by*

$$\hat{f}(\mathbf{x}) = \min_{\mathbf{a}, \mathbf{b}} \{ \langle \mathbf{a}, \mathbf{x} \rangle + \mathbf{b} : \langle \mathbf{a}, \mathbf{o} \rangle + \mathbf{b} \geq \mathbf{v}_o, \forall \mathbf{o} \in \mathcal{O} \}$$

is concave. Furthermore, $\hat{f} \leq \tilde{f}$ holds over \mathbb{R}^n for all increasing and concave functions \tilde{f} with $\tilde{f}(\mathbf{o}) \geq \mathbf{v}_o$.

The following proposition gives the tractable reformulation of Equation (19) based on Lemma 3. Our result shows that Equation (19) can be solved by a finite-dimensional linear programming, involving $2(R + E)$ variables and $4R + ER + L + E$ constraints without nonnegativity constraints.

Proposition 3. *Equation (19) can be reformulated as the minimization problem as shown in Equation*

(19), which is a finite linear programming problem when $h(\mathbf{x}, \boldsymbol{\xi})$ is affine in \mathbf{x} .

$$\min_{\mathbf{a}, \mathbf{b}, \mathbf{y}, \boldsymbol{\mu}} \sum_{e=1}^E p_e (a_e h(\mathbf{x}, \boldsymbol{\xi}_e) + b_e) \quad (19a)$$

$$\text{s.t. } y_{r+1} - y_r = \mu_r (\tau_{r+1} - \tau_r) \quad \forall r \in [R-1] \quad (19b)$$

$$y_{r+1} - y_r \geq \mu_{r+1} (\tau_{r+1} - \tau_r) \quad \forall r \in [R-2] \quad (19c)$$

$$0 \leq \mu_r \leq G \quad \forall r \in [R-1] \quad (19d)$$

$$\sum_{r=1}^{R-1} \mu_r \int_{\tau_r}^{\tau_{r+1}} \psi_l(\tau) d\tau \leq c_l \quad \forall l \in [L] \quad (19e)$$

$$y_0 = 0, y_R = 1 \quad (19f)$$

$$a_e \tau_r + b_e \geq y_r \quad \forall e \in [E], r \in [R] \quad (19g)$$

$$a_e \geq 0 \quad \forall e \in [E] \quad (19h)$$

Furthermore, the worst-case utility function is given by

$$u^{K^*}(\tau) = \frac{y_{r+1}^* - y_r^*}{\tau_{r+1} - \tau_r} \tau + \frac{\tau_{r+1} y_r^* - \tau_r y_{r+1}^*}{\tau_{r+1} - \tau_r}, \quad \tau \in [\tau_r, \tau_{r+1}] \text{ for } r = 1, \dots, R-1, \quad (20)$$

with $u^{K^*}(0) = 0$ and $u^{K^*}(R) = 1$, where \mathbf{y}^* is the optimal solution of Equation (19).

By Proposition 4, we can derive the worst-case utility function $u_{ij}^{K^*}$ for all experts and attributes $(i, j) \in \mathcal{I} \times \mathcal{J}$. The derived utility functions are then incorporated into the second-stage problem of OPA-PR to determine the optimal decision weights. Regarding the uncertainty of ranking parameters, it is easy to verify by the classical robust counterpart that the worst-case ranking parameters are achieved at $\underline{s}_{ij} = \min_{\tilde{s}_{ij} \in [\underline{s}_{ij}, \bar{s}_{ij}]} \{\tilde{s}_{ij}\}$ for all $(i, j) \in \mathcal{I} \times \mathcal{J}$.

Proposition 4. Given the worst-case utility function $u_{ij}^{K^*}$ for all experts and attributes $(i, j) \in \mathcal{I} \times \mathcal{J}$ and the ambiguity set $\mathcal{S} = \prod_{(i,j) \in \mathcal{I} \times \mathcal{J}} [\underline{s}_{ij}, \bar{s}_{ij}]$ for attribute rankings, the second-stage problem of OPA-PR admits the following equivalent reformulation:

$$\begin{aligned} & \max_{\mathbf{w}, z} z, \\ & \text{s.t. } RU_{ijr}^{K^*} z \leq t_i \underline{s}_{ij} w_{ijr}, \quad \forall (i, j, r) \in \mathcal{H}, \\ & \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R w_{ijr} = 1, \\ & w_{ijr} \geq 0, \quad \forall (i, j, r) \in \mathcal{H}, \end{aligned} \quad (21)$$

where $U_{ijr}^{K^*} = u_{ij}^{K^*}(R+1-r) / \sum_{r=1}^R u_{ij}^{K^*}(r)$ for all $(i, j, r) \in \mathcal{H}$.

The normalization of the elicited worst-case utility functions ensures consistency in the scale and direction of utility in OPA, where higher rankings correspond to lower assigned ranking parameters. Alternatively, instead of the normalization method in Proposition 4, one can compute the disutility first and then normalize it. However, this approach leads to a situation where the utility of the last-ranked alternative is zero. The optimal weights w_{ijr}^* are then mapped to w_{ijk}^* according to alternative rankings under attributes by experts, with Equation (5) calculating the final weights for experts, attributes, and alternatives.

Another way to understand the worst-case ranking parameters of second-stage problem of OPA-PR is through the derivation of OPA with uncertain parameters. Consider Equation (2) with uncertain ranking parameters:

$$\delta(\mathbf{w}, \zeta) = \begin{cases} \delta^1(w_{ijk}, w_{ijl}, \tilde{s}_{ij}) = t_i \tilde{s}_{ij} r_{ijk} (w_{ijk} - w_{ijl}) > 0, & \forall \tilde{s}_{ij} \in \mathcal{S}_{ij}, (i, j, k, l) \in \mathcal{X}^1, \\ \delta^2(w_{ijk}, \tilde{s}_{ij}) = t_i \tilde{s}_{ij} r_{ijk} (w_{ijk}) > 0, & \forall \tilde{s}_{ij} \in \mathcal{S}_{ij}, (i, j, k) \in \mathcal{X}^2. \end{cases}$$

Then, we have $\max_{\mathbf{w} \in \mathcal{W}} \min_{\zeta \in \mathcal{V}} \delta(\mathbf{w}, \zeta)$. Using the max-min method and variable substitution, the formulation for the second-stage problem of OPA-PR, which is conservative due to its max-min-min structure, is the same with Equation (21).

It is worth noting that when the worst-case utility function u_{ij}^{K*} for all experts and attributes $(i, j) \in \mathcal{I} \times \mathcal{J}$ reduce to rank order centroid weights, and the ambiguity set for attribute rankings are discrete scenario, OPA-PR recovers the OPA-R proposed by Mahmoudi et al. (2022a).

Theorem 2. *Given the elicited the worst-case utility function u_{ij}^{K*} and attribute ranking \underline{s}_{ij} for all experts and attributes $(i, j) \in \mathcal{I} \times \mathcal{J}$, the closed-form solution of second-stage problem of OPA-PR is*

$$z^{K*} = 1 / \left(\sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \frac{RU_{ijr}^{K*}}{t_i \underline{s}_{ij}} \right), \quad (22)$$

and

$$w_{ijr}^{K*} = (RU_{ijr}^{K*}) / \left(t_i \underline{s}_{ij} \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \frac{RU_{ijr}^{K*}}{t_i \underline{s}_{ij}} \right), \quad \forall (i, j, r) \in \mathcal{H}, \quad (23)$$

where $U_{ijr}^{K*} = u_{ij}^{K*} (R + 1 - r) / \sum_{r=1}^R u_{ij}^{K*} (r)$ for all $(i, j, r) \in \mathcal{H}$.

Corollary 2. *The optimal value z^{K*} of the second-stage problem of OPA-PR is constant for any utility function $u_{ij}^K \in \mathcal{U}_{ij}^K$ for all $(i, j) \in \mathcal{I} \times \mathcal{J}$, which is given by $z^{K*} = 1 / \left(R \sum_{i=1}^I \frac{1}{t_i} \left(\sum_{j=1}^J \frac{1}{\underline{s}_{ij}} \right) \right)$.*

By Corollary 2, z^{K*} depends solely on the expert rankings, the worst-case attribute rankings, and the number of alternatives, which means regardless of the elicited worst-case utility functions for all experts and attributes, the optimal value remains constant. It follows that the optimal weight disparity scalar

z^{K*} in Theorem 2 provides a general lower-bound reference for designing weight disparity scalars to other weight elicitation methods. Another implication of Corollary 2 is that, in OPA-PR, the robustness against parameter uncertainty means a tendency towards balance. This can be easily verified that the optimal value z^{K*} of OPA-PR is always less than that of OPA.

Corollary 3. *The weights of experts $W_i^{\mathcal{I}}$ for all $i \in \mathcal{I}$ and attributes $W_j^{\mathcal{J}}$ for all $j \in \mathcal{J}$ are independent of the worst-case utility function u_{ij}^{K*} of corresponding expert i and attribute j .*

Corollary 3 states that the worst-case utility function with risk-averse preferences only affects the weights assigned to alternatives, without influencing the weights of experts and attributes. This illustrates a property of risk preference independence of OPA-PR.

3.4. Error Bound on the Optimal Value

This section analyze the impact of the approximation of the ambiguity set of utility functions on the optimal value for the both stages to provide a theoretical basis for the application of PLA.

Compared to the first-stage error bound of OPA-PR, our primary interest lies in identifying which forms of partial preference information minimize the first-stage error, as this would inform the design of more effective preference elicitation strategies for DMs. The following proposition shows that the PLA of the first-stage problem in OPA-PR introduces no additional approximation errors when the utility functions in the ambiguity set are concave and the rankings on Θ corresponds to discrete outcomes.

Proposition 5. *If any $u \in \mathcal{U}$ is concave and $\psi_l(\tau)$ for $l = 1, \dots, L$ are step functions over Θ jumps at τ_r for $r = 1, \dots, R$, then $\rho^K = \rho$.*

The following proposition gives a equivalent representation of the PLA of the ambiguity sets for utility functions, which is obtained from the step-like approximation of the partial preference information in moment-type preference elicitation set.

Proposition 6. *Let $u \in \mathcal{U}$ be concave and u^K be the corresponding piecewise linear approximation defined as Equation (17). Then, the step-like approximation of ψ_l for $l = 1, \dots, L$, denoted as ψ_l^K , resulting from the discretization at τ_r for $r = 1, \dots, R$ such that $\psi_l^K(\tau_r) = \psi_l(\tau_r)$ is equivalent to the piecewise linear approximation of u .*

Propositions 5 and 6 provides insights for designing effective preference elicitation strategies in OPA-PR: we can avoid approximation errors by properly eliciting the DM's preferences with the step-like characteristic. Notably, as we discussed before, the partial preference information derived from deterministic utility comparison (Example 1) and stochastic lottery comparison with two lotteries (Example 2) is

a step function. The preference elicitation strategies of OPA-PR will be presented in Section 3.5, based on the above findings.

We next quantify the impact of the elicited worst-case utility functions with PLA in the first-stage problem on the optimal solution of the second-stage problem. To achieve this, it suffices to analyze the difference between u^{K*} and u^* and its impact on the optimal solution, as the PLA solely affects the structure of ambiguity set for utility functions, especially moment-type preference elicitation. It is notable that PLA may have impact on the second-stage problem even when no extra error is introduced in the first-stage problem. This is because the second-stage problem relies on the arguments from the first-stage minimization problem (i.e., u_{ij}^{K*} for all experts and attributes $(i, j) \in \mathcal{H}$), rather than the optimal value ρ^K . We first introduce the pseudo-metric for measuring the disparity between utility functions.

Definition 3. Let \mathcal{F} be a set of measurable functions defined over Θ . For any $u_1, u_2 \in \mathcal{U}$, the pseudo-metric between u_1 and u_2 is defined as

$$d_{\mathcal{F}}(u_1, u_2) := \sup_{f \in \mathcal{F}} |\langle f, u_1 \rangle - \langle f, u_2 \rangle|,$$

where $\langle \cdot \rangle$ denotes the Euclidean inner product.

It is easy to verify that $d_{\mathcal{F}}(u_1, u_2) = 0$ if and only if $\langle f, u_1 \rangle = \langle f, u_2 \rangle$ for all $f \in \mathcal{F}$. Based on the pseudo-metric, the following lemma provides the upper bound for the PLA errors.

Lemma 4. Let $\mathcal{F} := \{f = I_{\Theta}(\cdot)\}$, where $I_{\Theta}(\cdot)$ is the indicator function with domain Θ . For the optimal $u^* \in \mathcal{U}$, assume that u^* is G -Lipschitz continuous over Θ and u^{K*} is corresponding piecewise linear approximation defined as Equation (17). Then

$$d_{\mathcal{F}}(u^*, u^{K*}) = \sup_{t \in \Theta} |u^{K*}(t) - u^*(t)| \leq G, \quad (24)$$

where $G \geq 1/R$.

With Lemma 4, we can quantify the error bound on the optimal solution of the second-stage problem in OPA-PR originating from the PLA of the elicited worst-case utility functions.

Theorem 3. Assume that $\mathcal{F} := \{f = I_{\Theta}(\cdot)\}$ and $u^* \in \mathcal{U}$ is G -Lipschitz continuous over Θ and u^{K*} is corresponding piecewise linear approximation defined as Equation (17). Let w_{ijr}^* and w_{ijr}^{K*} denote the optimal weights of the second-stage problem of OPA-PR derived from $u_{ij}^*(R+1-r)$ and $u_{ij}^{K*}(R+1-r)$ for all $(i, j, r) \in \mathcal{H}$, respectively. Then, the optimal value z^{K*} (i.e., the optimal weight disparity) is constant and

$$|w_{ijr}^* - w_{ijr}^{K*}| \leq \frac{Rz^{K*}}{t_i s_{ij}} \left(\frac{2 \left(G + u_{ij}^{K*}(R+1-r) \right)}{R-1} - \frac{u_{ij}^{K*}(R+1-r)}{\sum_{r=1}^R u_{ij}^{K*}(r)} \right), \quad \forall (i, j, r) \in \mathcal{H}, \quad (25)$$

where z^{K*} is defined in Equation (22).

Notably, the error bound in Theorem 3 is not tight for concave utility functions, exhibiting maximum disparity when the utility functions approach risk-neutral linearity. The error bound depends on the value of u_{ij}^{K*} , especially the relation between $G + u_{ij}^{K*}(R+1-r)$ and 1. We can always restrict $G + u_{ij}^{K*}(R+1-r) \geq 1$ to $G + u_{ij}^{K*}(R+1-r) = 1$ to obtain a tighter error bound. For further details, we refer to the discussion on worst-case expected utility formulations biased toward risk neutrality in [Armbruster and Delage \(2015\)](#).

3.5. Preference Elicitation Strategy Design

This section focuses on the elicitation strategy for the DM's preference relevant to the first-stage problem in OPA-PR, while the ambiguity sets for attribute or expert rankings can be trivially generated from the maximum and minimum rankings. As discussed in Section 3.3.1, the way to elicit expert preferences directly influences the PLA errors of the ambiguity set for utility function. Thus, we utilize the stochastic pairwise lottery comparisons, which provide step-like preference information covering the deterministic utility comparisons, without introducing additional errors in the PLA as detailed in Propositions 5 and 6. Specifically, experts are asked to determine the certainty equivalent of pairwise lotteries:

$$Z_1 = \begin{cases} r_1, & \text{with probability } 1-p, \\ r_3, & \text{with probability } p, \end{cases} \quad \text{and} \quad Z_2 = r_2.$$

The question l is characterized by four parameters $r_1 \leq r_2 \leq r_3$ and p with the corresponding ψ_l given by

$$\psi_l(\tau) = (1-p)I_{[r_1, R]}(\tau) + pI_{[r_3, R]}(\tau) - I_{[r_2, R]}(\tau).$$

If the utility function is normalized such that $u(r_1) = 0$ and $u(r_3) = 1$, the question simplifies to determining whether $u(r_2) \geq p$ or not. Thus, the key is how to determine the probability p .

In the following, we utilize the random utility split scheme to select these four parameters. We first determine the number of questions L , from which the number of breakpoints is $3L + 2 \leq R$.

- Initialization: Set $l = 0$.
- Step 1: Choose r_1 and r_3 uniformly from the ranking set $\mathcal{R} = \{1, 2, \dots, R\}$ and r_2 uniformly from the rankings in $[r_1, r_3]$.
- Step 2: Normalize the utility function such that $u(r_1) = 0$ and $u(r_3) = 1$. Then, let $C := [\underline{C}, \bar{C}] := [\min_{u \in \mathcal{U}_N} u(r_2), \max_{u \in \mathcal{U}_N} u(r_2)]$ and $p = (\underline{C} + \bar{C}) / 2$. Taking the elicitation of the minimum \underline{C} as an

example, which is solved by the following minimization problem:

$$\begin{aligned}
& \min_{\mathbf{a} \succeq 0, \mathbf{b}} \sum_{r=1}^{R-1} (a_r r_2 + b_r) I_{(\tau_r, \tau_{r+1}]}(r_2), \\
& \text{s.t. } a_{r_1-1} r_1 + b_{r_1-1} = 0, \\
& \quad a_{r_3-1} r_3 + b_{r_3-1} = 1, \\
& \quad a_{r-1} \tau_r + b_{r-1} = a_r \tau_r + b_r, \quad \forall r \in [R-1], \\
& \quad a_{r+1} - a_r \leq 0, \quad \forall r \in [R-2], \\
& \quad \sum_{r=1}^{R-1} a_r \int_{\tau_r}^{\tau_{r+1}} \psi_l(\tau) d\tau \leq c_l, \quad \forall l \in [L], \\
& \quad a_r \leq G \quad \forall r \in [R].
\end{aligned}$$

Similarly, \bar{C} can be determined by maximizing the above problem.

- Step 3: Let $l = l + 1$ and $\psi_l(\tau) = (1 - p)I_{[r_1, R]}(\tau) + pI_{[r_3, R]}(\tau) - I_{[r_2, R]}(\tau)$. If $p \geq u(r_2)$, which is equivalent to $\int_0^\theta -\psi_l(\tau) du(\tau) \leq 0$, then the expert prefers Z_1 to Z_2 . We then include the constraint $\int_0^\theta -\psi_l(\tau) du(\tau) \leq 0$ in the ambiguity set \mathcal{U}_K . Otherwise, add $\int_0^\theta \psi_l(\tau) du(\tau) \leq 0$. Return to Step 1 until $l = L$.

Step 1 generates the lottery and certainty equivalent for pairwise comparisons. Step 2 determines the probability p at the midpoint of interval C , which means the true utility function value at r_2 lies in either the upper or lower half of interval C , reducing the ambiguity set for possible utility functions by half. Step 3 requires the expert to select the lottery and certainty equivalent to add the constraints. Notably, when the expert specifies the nominal utility function, the modulus G of Lipschitz continuity modulus can be derived from its first-order derivative.

Remark 2. It is important to acknowledge that preference inconsistencies may occur during the elicitation process, arising from factors such as violations of expected utility theory axioms or contamination of preference information (Guo et al., 2024). In Appendix A, we present extended formulations for three types of preference inconsistencies commonly discussed in literature (Guo et al., 2024; Wu et al., 2022): weight disparity error, utility outcome perturbation, and erroneous elicitation.

4. Robust Satisficing Extension

This section introduces the robust satisficing extension (OPA-PRS) of OPA-PR, especially the second-stage problem, to deal with the potential misspecification of the expert rankings provided by DM. Suppose that the true expert rankings $\tilde{t} \in \mathcal{T}$ are unknown and may vary from those provided by DM. By Corollary

2, a trade-off can be made by tolerating some loss in weight disparity performance to enhance robustness against the misspecification of expert rankings (Long et al., 2023). Thus, we introduce the following second-stage formulation for OPA-PRS:

$$\max_{\mathbf{w} \in \mathcal{W}, \eta \geq 0} \left\{ \sum_{i \in \mathcal{I}} \eta_i : f_{ijr}(\alpha z^{K^*}) - \underline{g}_{ijr}(w_{ijr}, \tilde{t}_i) \geq \eta_i \|\tilde{t}_i - t_i\|, \forall \tilde{t}_i \in \mathcal{T}_i, (i, j, r) \in \mathcal{H} \right\}, \quad (26)$$

where α represents the expected target level of weight disparity, z^{K^*} is the optimal value provided by Theorem 2, t_i is the nominal ranking of expert i for all $i \in \mathcal{I}$, $f_{ijr}(\alpha z^{K^*}) = \alpha RU_{ijr}^{K^*} z^{K^*}$, and $\underline{g}_{ijr}(w_{ijr}, \tilde{t}_i) = \tilde{t}_i \underline{s}_{ij} w_{ijr}$ for all $(i, j, r) \in \mathcal{H}$. The expected target level $\alpha z^{K^*} \leq z^{K^*}$ allows additional flexibility to improve the robustness against uncertainty originating from model misspecification. A violation of the target performance level occurs in OPA or OPA-PR when $\alpha RU_{ijr}^{K^*} z^{K^*} = f_{ijr}(\alpha z^{K^*}) > \underline{g}_{ijr}(w_{ijr}, \tilde{t}_i) = \tilde{t}_i \underline{s}_{ij} w_{ijr}$ for some $(i, j, r) \in \mathcal{H}$. Intuitively, OPA-PRS identifies the largest fragility of the weight system, measured by the largest magnitude of the target violations. As the total fragility $\sum_{i \in \mathcal{I}} \eta_i$ increases, the extent of the expected target violation grows accordingly. Notably, there is a connection between OPA-PRS and the modified formulation of generalized ordinal priority approach (GOPA) considering errors in weight disparity as proposed by Wang (2024a). While the modified GOPA formulation maximizes weight disparity within a permissible total error, OPA-PRS identifies the maximum total fragility within an acceptable weight disparity.

Remark 3. It is important to clarify that OPA-PRS should not be viewed as a relaxation of OPA-PR. For instance, let \mathcal{B} be the ambiguity set for expert rankings. The robust constraints indicate that

$$\begin{aligned} \mathbf{f}(\alpha z^{K^*}) - \mathbf{g}'(\mathbf{w}, \tilde{t}) &\succeq 0, & \forall \tilde{t} \in \mathcal{B}, \\ \mathbf{f}(\alpha z^{K^*}) - \mathbf{g}'(\mathbf{w}, \tilde{t}) &\succeq -\infty, & \forall \tilde{t} \in \mathcal{T} \setminus \mathcal{B}, \end{aligned}$$

whereas the robust satisficing constraints corresponding to the dual problem of OPA in Lemma 2 imply that

$$\begin{aligned} \mathbf{f}(\alpha z^{K^*}) - \mathbf{g}'(\mathbf{w}, \tilde{t}) &\succeq \eta \|\tilde{t} - t\|, & \forall \tilde{t} \in \mathcal{B}, \\ \mathbf{f}(\alpha z^{K^*}) - \mathbf{g}'(\mathbf{w}, \tilde{t}) &\succeq \eta \|\tilde{t} - t\|, & \forall \tilde{t} \in \mathcal{T} \setminus \mathcal{B}, \end{aligned}$$

for minimizing $\eta \geq 0$. Therefore, while OPA-PRS accepts some losses when $\tilde{t} \in \mathcal{B}$, it provides better protection against severe losses when $\tilde{t} \in \mathcal{T} \setminus \mathcal{B}$.

Proposition 7. Given the target performance level α , worst-case utility functions $u_{ij}^{K^*}(\cdot)$, nominal expert rankings t_i for all $i \in \mathcal{I}$, attribute rankings \underline{s}_{ij} for all experts and attributes $(i, j) \in \mathcal{I} \times \mathcal{J}$, and ambiguity set $\mathcal{T} = \times_{i \in \mathcal{I}} [t_i, \bar{t}_i] \ni \tilde{t}_i$ for all $i \in \mathcal{I}$, OPA-PRS in Equation (26) admits the equivalent reformulation in

Equation (27).

$$\begin{aligned}
& \min_{\mathbf{w}, \boldsymbol{\eta}, \varepsilon^1, \varepsilon^2} \sum_{i=1}^I \eta_i \\
& \text{s.t. } \alpha R U_{ijr}^{K^*} z^{K^*} \geq (\bar{t}_i - t_i) \varepsilon_i^1 - (\underline{t}_i - t_i) \varepsilon_i^2 + t_i \underline{s}_{ij} w_{ijr} \quad \forall (i, j, r) \in \mathcal{H} \\
& \quad \eta_i \geq \underline{s}_{ij} w_{ijr} + \varepsilon_i^2 - \varepsilon_i^1 \quad \forall (i, j, r) \in \mathcal{H} \\
& \quad \eta_i \geq -\underline{s}_{ij} w_{ijr} - \varepsilon_i^2 + \varepsilon_i^1 \quad \forall (i, j, r) \in \mathcal{H} \\
& \quad \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R w_{ijr} = 1 \\
& \quad w_{ijr} \geq 0, \eta_i \geq 0, \varepsilon_i^1 \geq 0, \varepsilon_i^2 \geq 0 \quad \forall (i, j, r) \in \mathcal{H}
\end{aligned} \tag{27}$$

Where $U_{ijr}^{K^*} = u_{ij}^{K^*}(R+1-r) / \sum_{r=1}^R u_{ij}^{K^*}(r)$ for all $(i, j, r) \in \mathcal{H}$ and $z^{K^*} = 1 / \left(\sum_{i=1}^I \sum_{j=1}^J \frac{R}{t_i \underline{s}_{ij}} \right)$.

As a new decision criterion for weight elicitation, it is also desired to characterize the fragility measure for weight disparity originating from OPA-PRS, which is not intuitive mathematically compared to that of OPA and OPA-PR (i.e., weight disparity). Let \mathcal{L} denote the space of bounded real-valued functions across all possible expert rankings in \mathcal{T} .

Definition 4 (Fragility Measure for Weight Disparity). A fragility measure for weight disparity of OPA-PRS with the nominal ranking t of single expert and fixed \mathbf{w} if and only if it has the following representation:

$$\begin{aligned}
& \phi \left(\mathbf{z}^\# - \mathbf{g}'(\mathbf{w}, \tilde{t}) \right) = \max \eta, \\
& \text{s.t. } z_{jr}^\# - g'_{jr}(w_{jr}, \tilde{t}) \geq \eta \|\tilde{t} - t\|, \quad \forall \tilde{t} \in \mathcal{T}, (j, r) \in \mathcal{J} \times \mathcal{R}, \\
& \quad \eta \leq 0,
\end{aligned} \tag{28}$$

where $z_{jr}^\# = \alpha R U_{jr}^{K^*} z^{K^*}$, $g'_{jr}(w_{jr}, \tilde{t}) = \frac{\tilde{t} \underline{s}_{jr}}{R U_{jr}^{K^*}} w_{jr}$ and $U_{jr}^{K^*} = u_j^{K^*}(R+1-r) / \sum_{r=1}^R u_j^{K^*}(r)$ for all $(j, r) \in \mathcal{J} \times \mathcal{R}$ with z^{K^*} and $U_{ijr}^{K^*}$ defined in Theorem 2.

With the above definition, we can rewrite Equation (26) into

$$\max_{\mathbf{w} \in \mathcal{W}} \sum_{i \in \mathcal{I}} \phi \left(\mathbf{z}_i^\# - \mathbf{g}'_i(\mathbf{w}_i, \tilde{t}_i) \right), \tag{29}$$

where $z_{ijr}^\# = R U_{ijr}^{K^*} z^{K^*}$ and $g'_{ijr}(w_{ijr}, \tilde{t}_i) = \frac{\tilde{t}_i \underline{s}_{ij}}{R U_{ijr}^{K^*}} w_{ijr}$ for all $(i, j, r) \in \mathcal{H}$.

Theorem 4. A fragility measure for weight disparity with the nominal expert ranking $t \in \mathcal{T}$ is a upper-semi-continuous functional $\phi : \mathcal{L} \rightarrow \mathbb{R}^-$ satisfying the following properties:

1. *Monotonicity:* If $\mathbf{g}'_2(\mathbf{w}, \tilde{t}) \succeq \mathbf{g}'_1(\mathbf{w}, \tilde{t})$ for all $\tilde{t} \in \mathcal{T}$, then $\phi(\mathbf{z}^\# - \mathbf{g}'_1(\mathbf{w}, \tilde{t})) \geq \phi(\mathbf{z}^\# - \mathbf{g}'_2(\mathbf{w}, \tilde{t}))$.

2. *Positive homogeneity:* For any $\lambda \geq 0$, $\phi(\lambda(\mathbf{z}^\# - \mathbf{g}'(\mathbf{w}, \tilde{t}))) = \lambda\phi(\mathbf{z}^\# - \mathbf{g}'(\mathbf{w}, \tilde{t}))$.
3. *Superadditivity:* $\phi((\mathbf{z}^\# - \mathbf{g}'_1(\mathbf{w}, \tilde{t})) + (\mathbf{z}^\# - \mathbf{g}'_2(\mathbf{w}, \tilde{t}))) \geq \phi(\mathbf{z}^\# - \mathbf{g}'_1(\mathbf{w}, \tilde{t})) + \phi(\mathbf{z}^\# - \mathbf{g}'_2(\mathbf{w}, \tilde{t}))$.
4. *Pro-robustness:* If $\mathbf{z}^\# - \mathbf{g}'(\mathbf{w}, \tilde{t}) \succeq 0$, then $\phi(\mathbf{z}^\# - \mathbf{g}'(\mathbf{w}, \tilde{t})) = 0$.

Monotonicity guarantees that the fragility measure cannot be lower than another if one weight disparity violation (i.e., values in the direction of infeasibility) is not smaller than the other across all misspecification scenarios of expert rankings. Positive homogeneity requires that the fragility measure scales proportionally with the scalar of disparity violations. Superadditivity implies that the collective fragility measure of the aggregated constraints should be greater than the sum of fragility measure considered separately. Pro-robustness states that if the weight disparity constraints are consistently feasible, then the fragility measure for weight disparity should be zero. Furthermore, these properties, along with upper semi-continuity, ensure the tractability of the OPA-PRS formulation.

5. Numerical Experiments

5.1. Case Description and Setup

In this section, we conduct numerical experiments on the proposed OPA-PR and OPA-PRS, applied to the emergency supplier selection problem (ESSP) in the context of the 7.21 mega-rainstorm disaster in Zhengzhou, China. We focus on the in-disaster scenario, which contrasts with pre-disaster conditions. Unlike the pre-disaster setting, the in-disaster phase demands rapid decision-making due to the high urgency and uncertainty that typically follow such events. This scenario challenges traditional supplier selection as the unpredictability and complexity of disasters often make preselected suppliers inadequate (Wang et al., 2022). In response, the in-disaster ESSP must adapt by relying on partial preference information and accommodating DMs' risk preferences, significantly influencing outcomes under these conditions. Overall, the selection of in-disaster ESSP to demonstrate applicability and effectiveness of OPA-PR and OPA-PRS aligns with the proposed decision framework. Meanwhile, the urgency and complexity of decision-making in this context provide an ideal testing ground.

There are ten emergency suppliers (labeled A1 to A10) available, each presenting unique characteristics suited for flood relief efforts. Some provide stable supply chains and rapid response capabilities at higher costs, while others leverage strategic locations and efficient transport networks to improve emergency effectiveness. Six attributes are identified to evaluate these suppliers: response speed (C1), delivery reliability (C2), geographic coverage (C3), operational sustainability (C4), collaborative experience and credibility (C5), and supply cost (C6) (Wang, 2024a). The decision process involves five experts from various departments (labeled E1 to E5), prioritized by their decision-making authority in the following

order: E5, E2, E1, E3, and E4. These experts provide rankings for each attribute and supplier, with the data presented in Online Supplemental Material.

The numerical experiments are structured into three distinct sections: the case test, the sensitivity test, and the comparison test. The case test is designed to demonstrate the implementation of our proposed approach and provide an interpretable explanation of the case results. The sensitivity test assesses the effect of parameter variations on the outcomes. The comparison test contrasts the results with those obtained using other mainstream benchmarking MADM methods.

5.2. Experiment Results

This section presents the case results for the proposed OPA-PR and OPA-PRS model using the collected data. As for OPA-PRS, we set the target performance level to 0.9, with results for other performance levels analyzed in sensitivity analysis. Figure 2 illustrates the worst-case utility of alternatives under various attributes based on partial preference information obtained through the scheme in Section 3.5, which is the first-stage outcome. The results indicate that the worst-case utility of ranked alternatives varies according to the experts' partial preference information despite experts responding to identical utility elicitation questionnaires for the same attributes. This highlights the need to customize DMs' preferences in decision-making processes. Additionally, the worst-case utility function exhibits concavity (a risk-averse attitude), contrasting with the original OPA model with rank order centroid weights (suggesting a quasi-risk-seeking attitude) (Wang, 2024a). Thus, the proposed approach extends beyond the single risk-seeking preference in OPA, which is less common in practice.

Figure 3 shows the optimal weights for experts, attributes, and alternatives of both OPA-PR (Figure 3 (a)-(c)) and OPA-PRS (Figure 3 (d)-(f)). As for OPA-PR, starting with expert weights, E5 is the most influential, holding the highest weight of 0.4258, followed by E2 with 0.2269. The weights for E1, E3, and E4 follow at 0.1494, 0.1099, and 0.0880, respectively. The weight results of OPA-PRS show the following rankings, from highest to lowest: E5 (0.3333), E1 (0.2125), E2 (0.1971), E3 (0.1311), E4 (0.1261). Overall, the differences in expert weights are smaller, aligning with the target weight disparity of 0.9 in OPA-PRS. Notably, the rankings of E1 and E2 have reversed. Upon examining the input data, we find that E1 has a nominal rank of 3, with a ranking ambiguity set of [2, 3], while E2 has a nominal rank of 2, with an ambiguity set of [2, 4]. This suggests that OPA-PRS, by considering both nominal rankings and their bounds, places E1 in a more favorable ranking range than E2. Regarding attributes, the results of OPA-PR show that response speed (C1) is identified as the most critical, with a weight of 0.2549. Collaborative experience and credibility (C5) are also highly valued, weighting 0.2206. The weights for delivery reliability (C2), geographic coverage (C3), and supply cost (C6) are relatively lower but closely matched at 0.1405, 0.1387, and 0.1349, respectively. Operational sustainability (C4) is considered the least critical, reflected

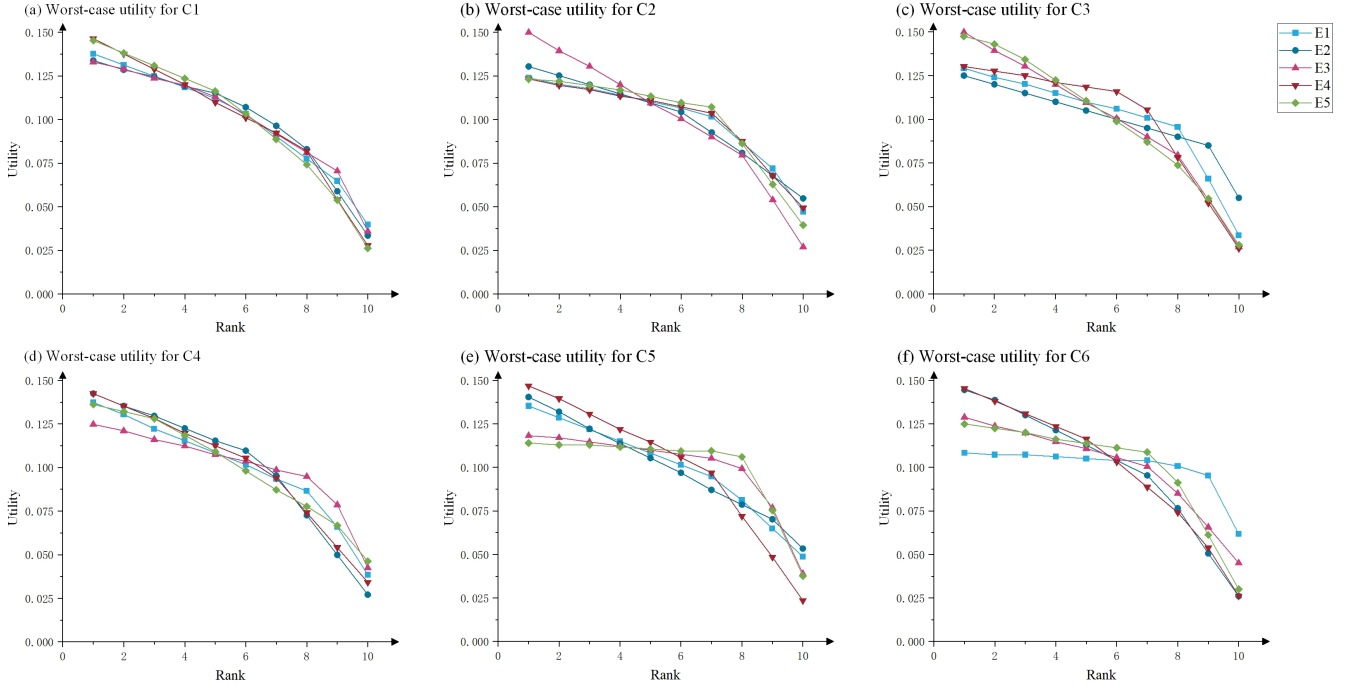


Figure 2: Worst-case utility for ranked alternatives under experts and attributes

by its lowest weight of 0.1104. The weight results of OPA-PRS show the following rankings, from highest to lowest: C1 (0.2634), C5 (0.1986), C3 (0.1618), C6 (0.1354), C2 (0.1339), and C4 (0.1068). OPA-PR and OPA-PRS are consistently ranked in C1, C5 and C4. For alternative weights, OPA-PR ranks A8, A5, A3, A9, A2, and A6 as the top six, with weights above 0.1000. In contrast, OPA-PRS ranks A8, A5, A6, A9, A2, and A3 as the top six. The results show that OPA-PR and OPA-PRS place A8 first, with A5 slightly trailing in second. A8 stands out in response speed (C1), operational sustainability (C4), and collaborative experience and credibility (C5), making it the most potent alternative. A5 complements A8 by excelling in delivery reliability (C2) and geographic coverage (C3). Based on these weights, A8 is the recommended alternative.

5.3. Sensitivity Analysis

This section employs sensitivity analysis to validate the effectiveness of OPA-PR and OPA-PRS. As for OPA-PR, we focus on the sensitivity of expert rankings as the validation object. The partial preference elicitation results and rankings of attributes and alternatives reflect true preferences and have significant practical implications for the interpretability of results. Sensitivity analysis of these parameters complicates interpretation due to the absence of a baseline for comparison. Conversely, outcome changes resulting from variations in expert rankings are more intuitive. Thus, a permutation approach is used to generate 120 potential scenarios to examine the impact of varying rankings on outcomes, with results presented in the box plots in Figure 4.

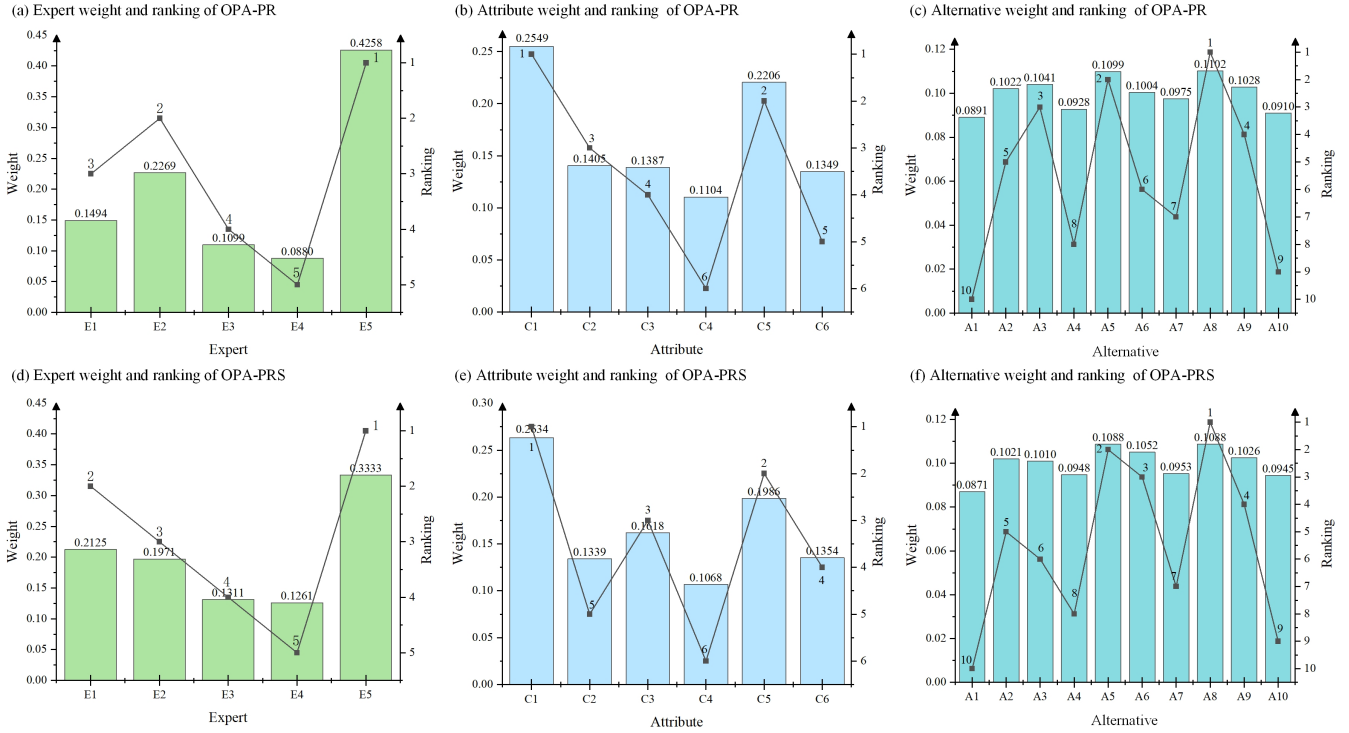


Figure 3: Optimal weights for experts, attributes, and alternatives of OPA-PR and OPA-PRS

Table 1 presents detailed weight results and their descriptive statistics indicators. Descriptive statistical analysis of expert weights indicates relative consistency among the mean value of five experts, averaging 0.2, with a minimum of 0.0835 and a maximum of 0.4480. This consistency is further supported by all other assessment metrics. It follows from the fact that the equal frequency of appearances by each expert across various ranking positions reinforces the consistency of the descriptive statistical results among the experts. Among the attributes, C1 has the highest mean value at 0.2917, followed by C3, C5, and C6 at 0.1777, 0.1551, and 0.1459, respectively. C2 and C4 have the lowest mean values at 0.1168 and 0.1129, respectively. Regarding skewness, most attributes, except C1, exhibit a right-skewed distribution, indicating that data primarily concentrates on the positive side, although C6 approaches symmetry. C2, C3, and C6 show higher kurtosis, while C1, C4, and C5 are relatively flat. Among these attributes, only C5 has a coefficient of variation of 0.2422, exceeding the acceptable threshold of 0.2, while the others exhibit comparatively stable dispersion. For the alternatives, the top four mean values are A8, A5, A6, and A2 at 0.1105, 0.1102, 0.1024, and 0.1017, respectively. The kurtosis of the alternatives ranges from $[-1.4156, -0.3803]$, suggesting a relatively flat distribution of weights. Furthermore, A1, A5, A7, A8, and A10 demonstrate positive skewness, indicating a tendency towards higher values, while the others exhibit negative skewness. The coefficient of variation for all alternatives remains below the specified threshold. Overall, the sensitivity analysis results for OPA-PR are relatively stable in terms of experts, attributes, and alternatives.

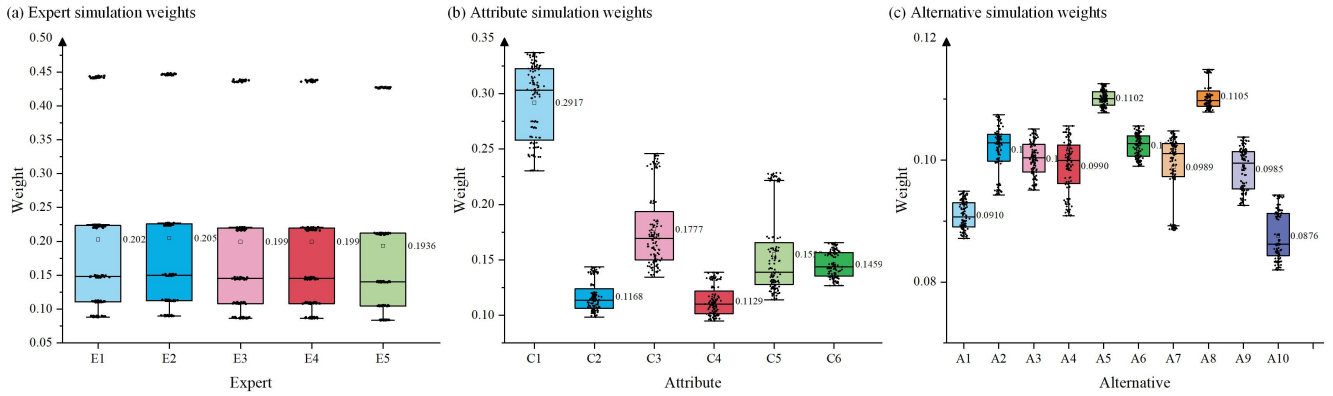


Figure 4: Box plots of the weight outcomes for sensitivity analysis in expert rankings of OPA-PR

For OPA-PRS, we examine the target performance level α , with values ranging from 0.7 to 1.0 in increments of 0.05. Figure 5 presents the weights and rankings of experts, attributes, and alternatives across different target performance levels. The results show that when $\alpha = 0.85$, the rankings of E1 and E2 are reversed: E2 dropped from second to third place, while E1 moved from third to second. Although E2 has a higher nominal ranking than E1 (E2 is second, E3 is third), its true ranking lies within a larger ambiguity set than E3 (E2's ambiguity set is [2, 4], while E3's is [2, 3]). This suggests that when the target weight disparity level is relatively small, the nominal ranking of E2 has a more significant impact than its ambiguity set of true rankings. Additionally, we observe shifts in the rankings of C2 and C6. For alternatives, rankings vary across different target performance levels. However, the optimal alternatives consistently remain A5 and A8. This further indicates that variations in weight disparities and constraint violations (i.e., model misspecification) would affect decision outcomes. The above results suggest that the OPA-PRS model is relatively stable and provides insights into the decision-making system.

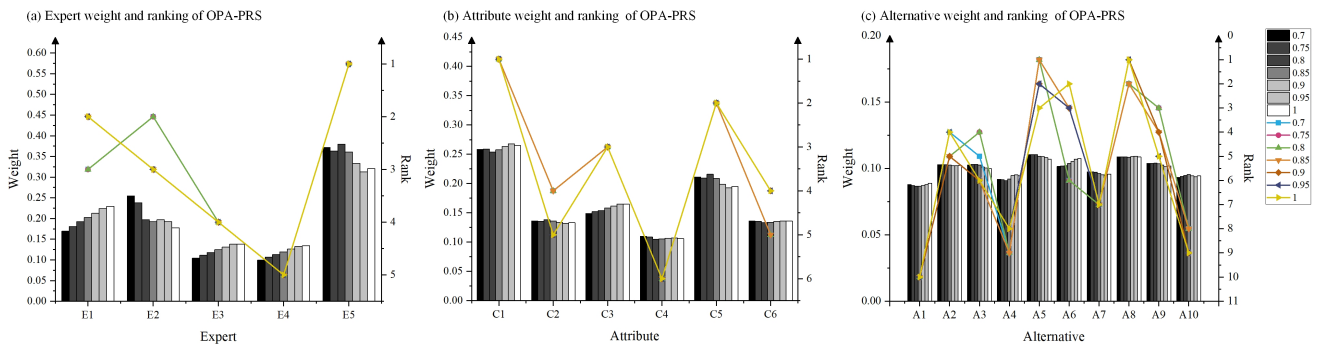


Figure 5: Sensitivity analysis result in target performance level of OPA-PRS

Table 1: Descriptive statistics of the weight outcomes for sensitivity analysis in expert rankings of OPA-PR

	Mean	Skewness	Kurtosis	Coefficient of variation	Min	Max
E1	0.2027	1.0984	-0.3284	0.6349	0.0882	0.4445
E2	0.2050	1.0956	-0.3326	0.6323	0.0897	0.4480
E3	0.1993	1.1026	-0.3221	0.6388	0.0863	0.4389
E4	0.1993	1.1026	-0.3221	0.6388	0.0863	0.4389
E5	0.1936	1.1097	-0.3116	0.6453	0.0835	0.4279
C1	0.2917	-0.3665	-1.3347	0.1157	0.2304	0.3369
C2	0.1168	0.7694	-0.6739	0.1151	0.0984	0.1436
C3	0.1777	0.8598	-0.5905	0.1928	0.1345	0.2458
C4	0.1129	0.6630	-0.8321	0.1162	0.0950	0.1388
C5	0.1551	1.0502	-0.3721	0.2422	0.1140	0.2284
C6	0.1459	0.1334	-1.3417	0.0791	0.1268	0.1654
A1	0.0910	0.0688	-1.3517	0.0252	0.0871	0.0949
A2	0.1017	-0.6915	-0.6099	0.0377	0.0943	0.1075
A3	0.1002	-0.1422	-0.9917	0.0278	0.0951	0.1051
A4	0.0991	-0.4508	-0.8995	0.0430	0.0909	0.1056
A5	0.1102	0.0234	-1.0451	0.0120	0.1078	0.1125
A6	0.1024	-0.1457	-1.2832	0.0186	0.0990	0.1056
A7	0.0989	-1.0455	-0.3803	0.0542	0.0885	0.1048
A8	0.1105	0.9572	-0.4835	0.0200	0.1079	0.1148
A9	0.0985	-0.1879	-1.3945	0.0350	0.0926	0.1038
A10	0.0876	0.2668	-1.4156	0.0446	0.0820	0.0943

5.4. Comparison Analysis

This section validates OPA-PR and OPA-PRS by comparing its alternative ranking outcomes with those of eight other methods, including ELECTRE II, MABAC, MACBETH, MAUT, MOORA, OPA, TOPSIS, and VIKOR. These benchmark methods represent classical MADM approaches encompassing multi-attribute utility theory-based methods (MAUT and MACBETH), distance-based methods (MABAC, MOORA, TOPSIS, and VIKOR), and ranking-based methods (ELECTRE II and OPA). These reflect the prevailing concepts in MADM and provide a broad and representative set of benchmarks for the comparative analysis. Notably, the benchmark methods require disparate input data, necessitating the acquisition of corresponding data sets. We adapt the outputs of OPA-PR to the input requirements of benchmark methods to simulate the decision-making scenarios under incomplete information; sourcing complete data from experts would escalate the decision-making complexity and potential inaccuracies. We aggregate the output weights from OPA-PR across experts to determine the attribute weights for alternatives, as detailed in Table 2. As per Corollary 2, these weights incorporate attribute-specific weight information, prompting us to assign equal attribute weights for all benchmark methods. We default to the most com-

monly used settings in the literature for additional parameters required by these methods. In particular, the utility function in MAUT adopts an exponential form, which is common in portfolio optimization studies considering risk preferences. This study then utilizes the Spearman correlation coefficient to assess the correlation between alternative ranking outcomes. A coefficient near 1 indicates a strong positive correlation, while a value nearing -1 denotes a pronounced negative correlation. A coefficient close to 0 implies no significant correlation.

Table 2: Input dataset for the benchmark methods

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
C1	0.0235	0.0237	0.0303	0.0261	0.0298	0.0260	0.0160	0.0318	0.0244	0.0234
C2	0.0092	0.0145	0.0141	0.0139	0.0161	0.0155	0.0170	0.0105	0.0152	0.0145
C3	0.0113	0.0172	0.0143	0.0156	0.0130	0.0137	0.0153	0.0133	0.0150	0.0100
C4	0.0100	0.0103	0.0092	0.0123	0.0120	0.0106	0.0092	0.0137	0.0123	0.0107
C5	0.0237	0.0233	0.0239	0.0112	0.0251	0.0174	0.0248	0.0254	0.0234	0.0223
C6	0.0114	0.0132	0.0122	0.0137	0.0140	0.0170	0.0152	0.0156	0.0125	0.0100

Table 3 presents the ranking outcomes of various MADM methods. The top two alternatives are focused on A5 and A8, with A8 leading in most cases, including OPA-PR, OPA-PRS, ELECTRE, MABAC, BACBETH, MAUT, and OPA. Among the remaining methods, A5 ranks first.

Table 3: Alternative ranking results of multiple MADM methods

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
OPA-PR	10	5	3	8	2	6	7	1	4	9
OPA-PRS	10	5	6	8	2	3	7	1	4	9
ELECTRE II	5	4	4	5	2	4	5	1	3	5
MABAC	10	4	7	8	2	5	6	1	3	9
MACBETH	10	4	7	8	2	5	6	1	3	9
MAUT	10	4	7	8	2	5	3	1	6	9
MOORA	10	4	6	8	1	5	7	2	3	9
OPA	9	5	3	7	2	8	4	1	6	10
TOPSIS	10	4	6	8	1	5	7	2	3	9
VIKOR	10	5	7	8	1	3	6	4	2	9

Figure 6 displays a correlation heatmap for OPA-PR, OPA-PRS, and various benchmark methods. Analysis reveals generally positive correlations between OPA-PR and the benchmarks, albeit with lower correlations of 0.7576 and 0.7697 with VIKOR and MAUT, respectively. Meanwhile, MAUT's correlations with other methods are moderately high, ranging from 0.7576 to 0.8909. Except for VIKOR and MAUT, the correlation values of OPA-PR are relatively consistent, lying between 0.8788 and 0.9313. OPA-PR is

shown to exert a dominance relation on correlation outcomes compared to the original OPA, reflecting its adoption of risk-averse preferences, in contrast to theta risk-seeking tendencies of OPA. This highlights the differing decision outcomes driven by risk-seeking versus risk-averse attitudes. Moreover, OPA-PRS outperforms OPA-PR in most cases, except in its correlation with OPA. Thus, incorporating model misspecification in OPA-PRS significantly enhances the performance of OPA-PR.

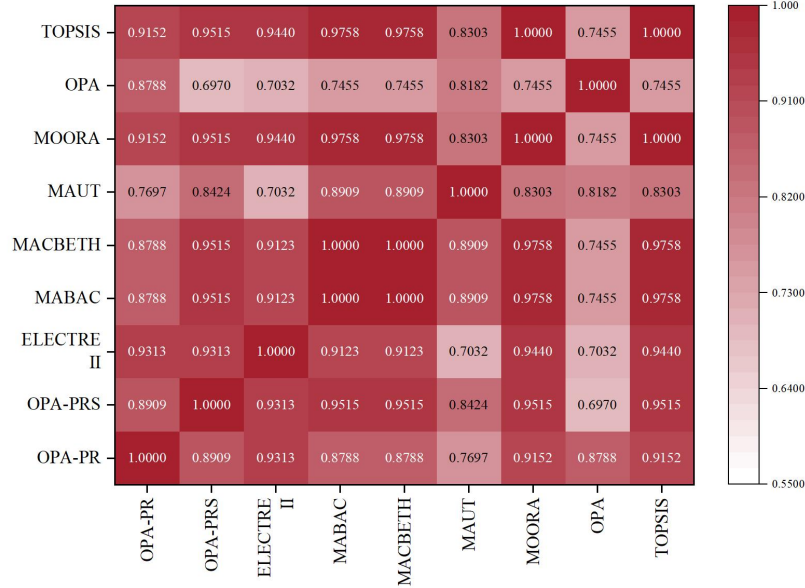


Figure 6: Correlation heat map of alternative rankings of multiple MADM methods

6. Concluding Remarks

In this study, we derive the equivalent reformulation of OPA and its properties of closed-form solution and decomposability. We reveal from the underlying that OPA optimizes weight disparities based on rank-order centroid weights within a normalized weight space, incorporating experts, attributes, and alternatives, while accounting for expert preferences. Motivated by this observation, we propose OPA-PR to counter the ambiguity within the ranking parameters and utility functions, followed by OPA-PRS to account for model misspecification and the above ambiguity. OPA-PR utilizes a two-stage optimization framework: in the first stage, it elicits the worst-case utility functions from the ambiguity sets of utility preferences, and in the second stage, it optimizes decision weights under the worst-case ranking parameters within support functions and the elicited worst-case utility functions. Given the difficulty of solving the original first-stage problem, we suggest utilizing PLA to obtain the worst-case utility functions. Subsequently, we present the tractable reformulation of the first and second stage problems, further deriving the properties of closed-form solution, invariance of optimal weight disparity scalar, and risk preference independence. We demonstrate that, with proper preference elicitation, PLA incurs no error in the objective

value (i.e., the weight disparity scalar) of first-stage problem. Additionally, we provide the error bound for the second-stage problem resulting from the PLA of the elicited worst-case utility functions, which forms theoretical basis for PLA and the subsequent preference elicitation strategy design. We further propose OPA-PRS and a corresponding decision criterion (i.e., fragility measure of weight disparity) to address scenarios where ranking parameters are misspecified. The proposed fragility measure of weight disparity has the properties of upper semi-continuity, monotonicity, positive homogeneity, superadditivity, and pro-robustness, which guarantees the computational tractability of OPA-PRS. Numerical experiments confirm the effectiveness of the proposed approach.

We emphasize that the conclusions related to the application are drawn from a limited context. Further applications in diverse scenarios are necessary to validate the effectiveness of the proposed approach. Additionally, we only consider the most general ambiguity set for the ranking parameter uncertainty, where experts directly provide the maximum and minimum rankings. Future research could explore tractable reformulations of OPA-PR and OPA-PRS by incorporating different types of ambiguity sets for ranking parameters, such as norm, polyhedral, and budget-based ambiguity sets. A key question is how to elicit expert preferences in designing the penalty levels for ambiguity sets. Under the artificial intelligence paradigm, integrating advanced operations research techniques such as distributionally robust optimization and stochastic contextual optimization shows promise for OPA-PR and OPA-PRS. Finally, we highlight the importance of future research on extending OPA under model misspecifications, which offers insightful implications for real-world decision-making.

Author Contributions

Renlong Wang: Conceptualization, Investigation, Methodology, Writing-original draft, Formal analysis, Validation, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability

Data will be made available on request.

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Appendix A. Extended Formulation Considering Preference Inconsistency

In this section, we discuss the potential preference inconsistency for the preference elicitation process, which may arise from factors such as misalignment with expected utility theory axioms or contaminated preference information (Guo et al., 2024). We examine three interpretable types of preference inconsistency, including weight disparity error, utility outcome perturbation, and erroneous elicitation, along with their corresponding modified formulations.

First, we assume there are errors in weight disparity, indicating that the optimized weight disparity scalar does not satisfy all weight disparity constraints. We can evaluate $\mathbf{g}(\mathbf{w}, \boldsymbol{\zeta})$ as $\mathbf{g}(\mathbf{w}, \boldsymbol{\zeta}) + \boldsymbol{\gamma}$ for a perturbation $\boldsymbol{\gamma} \in \mathbb{R}_+^{I \times J \times R}$ with a tolerated total error Γ in the second-stage problem of OPA-PR. Thus, we can relax the constraints $\max_{\boldsymbol{\zeta} \in \mathcal{V}} \{\mathbf{f}(z, \mathbf{u}^*) - \mathbf{g}(\mathbf{w}, \boldsymbol{\zeta})\} \preceq 0$ to

$$\begin{aligned} Ru_{ij}^*(r)z &\leq \gamma_{ijr} + \min_{\tilde{s}_{ij} \in \mathcal{S}_{ij}} \{t_i \tilde{s}_{ij} w_{ijr}\}, & \forall (i, j, r) \in \mathcal{H}, \\ \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \gamma_{ijr} &\leq \Gamma, & \\ \gamma_{ijr} &\geq 0, & \forall (i, j, r) \in \mathcal{H}, \end{aligned} \tag{A.1}$$

where $\boldsymbol{\gamma}$ is treated as a decision variable. The tolerated total error can be designed based on the optimal value of z^{K^*} from the original formulation of OPA-PR.

Next, we consider perturbations in elicited utility outcomes. In this scenario, the rankings of alternatives r can be adjusted to $r + \gamma_{ijr}$ for all $(i, j, r) \in \mathcal{H}$ in the second-stage problem of OPA-PR, where $\gamma_{ijr} \in (-1, 1)$ are specified parameters. Thus, the constraints $\max_{\boldsymbol{\zeta} \in \mathcal{V}} \{\mathbf{f}(z, \mathbf{u}^*) - \mathbf{g}(\mathbf{w}, \boldsymbol{\zeta})\} \preceq 0$ can be relaxed to

$$Ru_{ij}^*(r + \gamma_{ijr})z \leq \min_{\tilde{s}_{ij} \in \mathcal{S}_{ij}} \{t_i \tilde{s}_{ij} w_{ijr}\}, \quad \forall (i, j, r) \in \mathcal{H}. \tag{A.2}$$

Finally, when addressing erroneous elicitation, we can selectively relax a portion of the preference elicitation constraints in the second-stage problem of OPA-PR. In this case, we assume that at least $1 - \gamma_{ij}$ of the R dominance relations under each expert and attribute are correct, implying that expert i can give incorrect responses in at most $\gamma_{ij}R$ relations on attribute j . We introduce binary variables ϑ_{ijr} for all $(i, j, r) \in \mathcal{H}$, taking the value of 1 if expert i is incorrect about the ranking relation r on attribute j . Thus, the constraints $\max_{\boldsymbol{\zeta} \in \mathcal{V}} \{\mathbf{f}(z, \mathbf{u}^*) - \mathbf{g}(\mathbf{w}, \boldsymbol{\zeta})\} \preceq 0$ can be relaxed to

$$\begin{aligned} Ru_{ij}^*(r)z &\leq \vartheta_{ijr}M \min_{\tilde{s}_{ij} \in \mathcal{S}_{ij}} \{t_i \tilde{s}_{ij} w_{ijr}\}, & \forall (i, j, r) \in \mathcal{H}, \\ (1 - \vartheta_{ijr})M + Ru_{ij}^*(r)z &\leq \min_{\tilde{s}_{ij} \in \mathcal{S}_{ij}} \{t_i \tilde{s}_{ij} w_{ijr}\}, & \forall (i, j, r) \in \mathcal{H}, \\ \sum_{r=1}^R \vartheta_{ijr} &\leq \gamma_{ij}R, & \forall (i, j) \in \mathcal{I} \times \mathcal{J}, \end{aligned} \tag{A.3}$$

where M is a large constant.

Appendix B. Proofs for Section 2

PROOF OF LEMMA 1. We will show Lemma 1 by contradiction. Denote z^* the optimal value of z in Equation (4). Suppose that there exists σ_{ijk} such that $t_i s_{ij} r_{ijk}(w_{ijk} - w_{ijl}) = z^* + \sigma_{ijk}$ for any $(i, j, k, l) \in \mathcal{X}^1$ and $t_i s_{ij} r_{ijk}(w_{ijk}) = z^* + \sigma_{ijk}$ for any $(i, j, k) \in \mathcal{X}^2$, which means the constraints in Equation (4) are not all active. If $\exists \sigma_{ijk} < 0$, then there exists $t_i s_{ij} r_{ijk}(w_{ijk} - w_{ijl}) = \bar{z} < z^*$ or $t_i s_{ij} r_{ijk}(w_{ijk}) = \bar{z} < z^*$, which contradicts the optimality of z^* . Thus, we have $\sigma_{ijk} \geq 0$ for all $(i, j, k) \in \mathcal{Y}$ and there must be at least one σ_{ijk} such that $t_i s_{ij} r_{ijk}(w_{ijk} - w_{ijl}) = z^*$ or $t_i s_{ij} r_{ijk}(w_{ijk}) = z^*$.

We will present the following discussion under the mapping from the alterantive index k to the ranking index r . By cumulative sum of the last r terms in ascending order, we have

$$w_{ijr}^* = \frac{1}{t_i s_{ij}} \left(\sum_{h=r}^R \frac{1}{h} \right) (z^* + \sigma_{ijr}), \quad \forall (i, j, r) \in \mathcal{H}.$$

Substituting into the normalized constraints yields

$$\begin{aligned} \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R w_{ijr}^* = 1 &\Leftrightarrow z^* \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \left(\frac{1}{t_i s_{ij}} \sum_{h=r}^R \frac{1}{h} \right) + \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \left(\frac{\sigma_{ijr}}{t_i s_{ij}} \sum_{h=r}^R \frac{1}{h} \right) = 1 \\ &\Leftrightarrow z^* = \left(1 - \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \left(\frac{\sigma_{ijr}}{t_i s_{ij}} \sum_{h=r}^R \frac{1}{h} \right) \right) / \left(\sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \left(\frac{1}{t_i s_{ij}} \sum_{h=r}^R \frac{1}{h} \right) \right). \end{aligned}$$

For some $(i', j', r') \in \mathcal{H}$ such that $\sigma_{i'j'r'} > 0$, we can always select $\epsilon_{i'j'r'} > 0$ sufficiently small to ensure

$$w_{i'j'r'}^* = \frac{1}{t_{i'} s_{i'j'}} \left(\sum_{h=r'}^R \frac{1}{h} \right) (z^* + (\sigma_{i'j'r'} - \epsilon_{i'j'r'})),$$

and for all $(i, j, r) \setminus (i', j', r') \in \mathcal{H}$,

$$w_{ijr}^* = \frac{1}{t_i s_{ij}} \left(\sum_{h=r}^R \frac{1}{h} \right) \left(\left(z^* + \left(\epsilon_{i'j'r'} \sum_{h=r'}^R \frac{1}{h} \right) / \left(t_{i'} s_{i'j'} \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \left(\frac{1}{t_i s_{ij}} \sum_{h=r}^R \frac{1}{h} \right) \right) \right) + \sigma_{ijr} \right),$$

where

$$z^* + (\sigma_{i'j'r'} - \epsilon_{i'j'r'}) \geq z^* + \left(\epsilon_{i'j'r'} \sum_{h=r'}^R \frac{1}{h} \right) / \left(t_{i'} s_{i'j'} \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \left(\frac{1}{t_i s_{ij}} \sum_{h=r}^R \frac{1}{h} \right) \right).$$

Let

$$z^{\bar{*}} = z^* + \left(\epsilon_{i'j'r'} \sum_{h=r'}^R \frac{1}{h} \right) / \left(t_{i'} s_{i'j'} \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \left(\frac{1}{t_i s_{ij}} \sum_{h=r}^R \frac{1}{h} \right) \right).$$

For any $(i, j, r) \in \mathcal{H}$ with $\sigma_{ijr} > 0$, we can always identify $\epsilon_{ijr} > 0$ such that $z^{\bar{*}} > z^*$ satisfies all constraints in Equation (4) until $\sigma_{ijr} = 0$ for all $(i, j, r) \in \mathcal{H}$, which contradicts the optimality of z^* . Thus, all constraints in Equation (4) are active at optimality. \square

PROOF OF LEMMA 2. Consider the dual form of the reformulated OPA in Equation (6), with dual variables $\gamma \in \mathbb{R}^{I \times J \times R}$ and $\gamma_0 \in \mathbb{R}$ corresponding to the inequality and equality constraints, respectively. Using Lagrange duality, we derive the Lagrange function of Equation (6):

$$\begin{aligned} & L(z, w_{111}, \dots, w_{IJR}, \gamma_0, \gamma_{111}, \dots, \gamma_{IJR}) \\ &= z + \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \gamma_{ijr} \left(t_i s_{ij} w_{ijr} - \left(\sum_{h=r}^R \frac{1}{h} \right) z \right) + \gamma_0 \left(1 - \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R w_{ijr} \right) \\ &= \left(1 - \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \left(\sum_{h=r}^R \frac{1}{h} \right) \gamma_{ijr} \right) z + \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R (t_i s_{ij} \gamma_{ijr} - \gamma_0) w_{ijr} + \gamma_0. \end{aligned}$$

The dual problem of Equation (6) is expressed as

$$\begin{aligned} & \min_{\gamma, \gamma_0} \gamma_0, \\ & \text{s.t. } t_i s_{ij} \gamma_{ijr} - \gamma_0 \leq 0, \quad \forall (i, j, r) \in \mathcal{H}, \\ & \quad \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \left(\sum_{h=r}^R \frac{1}{h} \right) \gamma_{ijr} = 1, \\ & \quad \gamma_{ijr} \geq 0, \quad \forall (i, j, r) \in \mathcal{H}. \end{aligned}$$

Let $v_{ijr} = \left(\sum_{h=r}^R \frac{1}{h} \right) \gamma_{ijr}$ for all $(i, j, r) \in \mathcal{H}$. We have

$$\begin{aligned} & \min_{\gamma, \gamma_0} \gamma_0, \\ & \text{s.t. } t_i s_{ij} v_{ijr} \leq \left(\sum_{h=r}^R \frac{1}{h} \right) \gamma_0, \quad \forall (i, j, r) \in \mathcal{H}, \\ & \quad \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R v_{ijr} = 1, \\ & \quad \gamma_{ijr} \geq 0, \quad \forall (i, j, r) \in \mathcal{H}. \end{aligned}$$

By strong duality, we have $\gamma_0^* = z^*$. Thus, by symmetric argument as in the proof of Lemma 1, the dual problem shares the same solution set with the primal problem. \square

PROOF OF THEOREM 1. The reformulated OPA in Equation (6) is a typical linear programming problem. Thus, employing the Lagrange multiplier method, we have

$$L(z, w_{ijr}, \alpha, \beta_{ijr}) = z - \alpha \left(\sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R w_{ijr} - 1 \right) - \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \beta_{ijr} \left(t_i s_{ij} w_{ijr} - \left(\sum_{h=r}^R \frac{1}{h} \right) z \right). \quad (\text{B.1})$$

By Lemma 1, we always have

$$\frac{\partial L(z, w_{ijr}, \alpha, \beta_{ijr})}{\partial \alpha} = \frac{\partial L(z, w_{ijr}, \alpha, \beta_{ijr})}{\partial \beta_{ijr}} = 0, \quad \forall (i, j, r) \in \mathcal{H}.$$

These yields

$$w_{ijr} = \frac{1}{t_i s_{ij}} \left(\sum_{h=r}^R \frac{1}{h} \right) z, \quad \forall (i, j, r) \in \mathcal{H}, \quad (\text{B.2})$$

and

$$\sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R w_{ijr} = 1. \quad (\text{B.3})$$

Substituting Equation (B.2) into Equation (B.3) yields

$$z^* = 1 / \left(R \left(\sum_{p=1}^I \frac{1}{p} \right) \left(\sum_{q=1}^J \frac{1}{q} \right) \right),$$

and

$$w_{ijr}^* = \left(\sum_{h=r}^R \frac{1}{h} \right) / \left(t_i s_{ij} R \left(\sum_{p=1}^I \frac{1}{p} \right) \left(\sum_{q=1}^J \frac{1}{q} \right) \right), \quad \forall (i, j, r) \in \mathcal{H}.$$

which is the close-form solution of OPA. \square

PROOF OF COROLLARY 1. By the close-form solution of OPA, we can easily get

$$w_{ijr}^* = \frac{\frac{1}{R} \sum_{h=r}^R \frac{1}{h}}{t_i s_{ij} \left(\sum_{p=1}^I \frac{1}{p} \right) \left(\sum_{q=1}^J \frac{1}{q} \right)} = v_r^{ROC} v_s^{RR} v_t^{RR}, \quad \forall (i, j, r) \in \mathcal{H},$$

and

$$w_{is}^* = \sum_{r=1}^R \frac{\frac{1}{R} \sum_{h=r}^R \frac{1}{h}}{t_i s_{ij} \left(\sum_{p=1}^I \frac{1}{p} \right) \left(\sum_{q=1}^J \frac{1}{q} \right)} = v_s^{RR} v_t^{RR}, \quad \forall (i, s) \in \mathcal{I} \times \{1, \dots, J\}.$$

It follows that the rank order centroid weight is a special case of OPA for determining the weights of alternatives when the importance of experts and attributes is equal. The rank reciprocal weight is a special case of OPA for the weights of attributes when the importance of experts is equal. \square

Appendix C. Proofs for Section 3

PROOF OF PROPOSITION 2. Given that $\psi_l(\tau)$ on $\tau \in \Theta$ is a step function with jumps at τ_r for $r = 1, \dots, R$, we have

$$\int_0^\theta \psi_l(\tau) du(\tau) = \sum_{r=2}^R \psi_l(\tau'_r) (u(\tau_r) - u(\tau_{r-1})) = \int_0^\theta \psi_l(\tau) du^K(\tau),$$

where $\tau'_r \in [\tau_{r-1}, \tau_r]$ and the last equality follows from the fact that the integral only involves the value of u at these jumps. \square

PROOF OF PROPOSITION 3. We first divide the set of utility functions based on their values at rankings on Θ , denoted as $\mathcal{U}(y) := \{u : u(\tau_r) = y_r \text{ for all } r \in [R]\}$. Then, we have $\mathcal{U}(y) \cap \mathcal{U}_{\text{conc}} \neq \emptyset$, which yields Equations (19b) and (19c). Equations (19b) and (19c) express the concavity property by the first order

condition, $y_{r+1} - y_r = \mu_r(\tau_{r+1} - \tau_r)$ and $\mu_{r+1} \leq \mu_r$ for all $r \in [R-1]$, where μ_r represents the subgradient at ranking r . The left-hand side of Equation (19d) characterizes monotonic increasing properties of utility function, while the right-hand side represents the Lipschitz continuity with the modulus being bounded by G . Furthermore, $\mathcal{U}(y) \cap \mathcal{U}_{\text{nor}} \neq \emptyset$, which are represented by Equation (19f). Equation (19e) details the elicited moment-type preference information for each ranking points, which is derived from the fact that $du(\tau_r) = \mu_r$ for all $r \in [R-1]$. Since $u_{ij}^K(h(\mathbf{x}, \boldsymbol{\xi}_e))$ in the objective function is concave, non-decreasing, and affine in \mathbf{x} , we can apply the support function for increasing concave functions, as outlined in Lemma 3, to approximate $u_{ij}^K(h(\mathbf{x}, \boldsymbol{\xi}_e))$. This leads to the objective function formulation in Equation (19a) and the constraints in Equations (19g) and (19h). \square

PROOF OF THEOREM 2. This follows the symmetric argument as the proof of Lemma 1 and Theorem 1. We first show that all inequality constraints are active when Equation (19) achieving optimal, and then apply the Lagrange multiplier method to obtain the closed-form solution. \square

PROOF OF COROLLARY 2. By the closed-form solution in Theorem 2, we observe that

$$z^* = 1 \left/ \left(\sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R \frac{RU_{ijr}^*}{t_i \underline{s}_{ij}} \right) \right. = 1 \left/ \left(\sum_{i=1}^I \sum_{j=1}^J \frac{R \sum_{r=1}^R U_{ijr}^*}{t_i \underline{s}_{ij}} \right) \right. = 1 \left/ \left(\sum_{i=1}^I \sum_{j=1}^J \frac{R}{t_i \underline{s}_{ij}} \right) \right.,$$

where the last equality follows from $\sum_{r=1}^R U_{ijr}^* = \sum_{r=1}^R u_{ij}^*(R+1-r) / \sum_{r=1}^R u_{ij}^*(r) = 1$ for all $(i, j) \in \mathcal{I} \times \mathcal{J}$. \square

PROOF OF COROLLARY 3. By Corollary 2, the optimal objective value z^* is constant for any $u \in \mathcal{U}$. By Theorem 2, we have

$$w_{ij}^* = \sum_{r=1}^R w_{ijr}^* = \sum_{r=1}^R \frac{RU_{ijr}^* z^*}{t_i \underline{s}_{ij}} = \frac{Rz^* \sum_{r=1}^R U_{ijr}^*}{t_i \underline{s}_{ij}} = \frac{Rz^*}{t_i \underline{s}_{ij}},$$

which implies the weights w_{ij}^* of attribute $j \in \mathcal{I}$ under expert $i \in \mathcal{J}$ is independent of U_{ijr}^* . It follows that the weights of experts and attributes, given by $W_i^{\mathcal{I}} = \sum_{j=1}^J w_{ij}^*$ for all $i \in \mathcal{I}$ and $W_j^{\mathcal{J}} = \sum_{i=1}^I w_{ij}^*$ for all $j \in \mathcal{J}$, respectively, are independent of U_{ijr}^* . \square

PROOF OF PROPOSITION 5. Since $\mathcal{U}^K \subset \mathcal{U}$, then $\rho^K = \min_{u \in \mathcal{U}^K} \mathbb{E}_{\mathbb{P}}[u(h(\mathbf{x}, \boldsymbol{\xi}))] \geq \min_{u \in \mathcal{U}} \mathbb{E}_{\mathbb{P}}[u(h(\mathbf{x}, \boldsymbol{\xi}))] = \rho$. It suffices to show that $\rho^K \leq \rho$. Let σ be a sufficiently small positive number such that $\mathbb{E}_{\mathbb{P}}[u^\sigma(h(\mathbf{x}, \boldsymbol{\xi}))] \leq \rho + \sigma$ with $u^\sigma \in \mathcal{U}$. By Proposition 2, since ψ_l for $l = 1, \dots, L$ are step functions over Θ , there exists a piecewise linear concave function $u^{K\sigma} \in \mathcal{U}^K$ such that $u^{K\sigma} \leq u^\sigma$ for all $\tau_r \in \Theta$, which implies $\mathbb{E}_{\mathbb{P}}[u^{K\sigma}(h(\mathbf{x}, \boldsymbol{\xi}))] \leq \mathbb{E}_{\mathbb{P}}[u^\sigma(h(\mathbf{x}, \boldsymbol{\xi}))]$. Therefore, we have $\rho^K = \min_{u \in \mathcal{U}^K} \mathbb{E}_{\mathbb{P}}[u(h(\mathbf{x}, \boldsymbol{\xi}))] \leq \mathbb{E}_{\mathbb{P}}[u^{K\sigma}(h(\mathbf{x}, \boldsymbol{\xi}))] \leq \mathbb{E}_{\mathbb{P}}[u^\sigma(h(\mathbf{x}, \boldsymbol{\xi}))] \leq \rho + \sigma$. Since σ is arbitrarily small, it follows that $\rho^K \leq \rho$. \square

PROOF OF PROPOSITION 6. Given the piecewise linear approximated u^K with jumps at τ_r for $r = 1, \dots, R$, we have

$$\begin{aligned} \int_0^\theta \psi_l(\tau) du_N(\tau) &= \sum_{r=1}^{R-1} \int_{\tau_r}^{\tau_{r+1}} \psi_l(\tau) du_N(\tau) = \sum_{r=1}^{R-1} \frac{u(\tau_{r+1}) - u(\tau_r)}{\tau_{r+1} - \tau_r} \int_{\tau_r}^{\tau_{r+1}} \psi_l(\tau) d\tau \\ &= \sum_{r=1}^{R-1} (u(\tau_{r+1}) - u(\tau_r)) \frac{\int_{\tau_r}^{\tau_{r+1}} \psi_l(\tau) d\tau}{\tau_{r+1} - \tau_r} = \sum_{r=1}^{R-1} (u(\tau_{r+1}) - u(\tau_r)) \psi_l(\tau'_{r+1}), \end{aligned}$$

where $\tau'_{r+1} \in [\tau_r, \tau_{r+1}]$ and $\psi_l(\tau'_{r+1})$ is constant. On the other hand, consider the step-like approximation ψ_l^K for any $l = 1, \dots, L$, which is discrete step function with jumps at τ_r for $r = 1, \dots, R$. Then, we have

$$\int_0^\theta \psi_l^K(\tau) du(\tau) = \sum_{r=1}^{R-1} \int_{\tau_r}^{\tau_{r+1}} \psi_l^K(\tau) du(\tau) = \sum_{r=1}^{R-1} \int_{\tau_r}^{\tau_{r+1}} \psi_l(\tau'_{r+1}) du(\tau) = \sum_{r=1}^{R-1} \psi_l(\tau'_{r+1}) (u(\tau_{r+1}) - u(\tau_r)).$$

Thus, the step-like approximation of ψ and piecewise linear approximation of u have the same effect. \square

PROOF OF LEMMA 4. Given $\mathcal{F} := \{f = I_{[0,\theta]}(\cdot)\}$, $d_{\mathcal{F}}(u_1, u_2)$ is well-defined for any $u_1, u_2 \in \mathcal{U}$. Since u^* is Lipschitz continuous with modulus G and increasing on Θ , we have

$$0 \leq \frac{u^*(\tau_r) - u^*(\tau_{r-1})}{\tau_r - \tau_{r-1}} \leq G, \quad r = 2, \dots, R.$$

Then, by Proposition 2, for any $\tau \in [\tau_{r-1}, \tau_r]$, we have

$$\begin{aligned} |u^{K*}(\tau) - u^*(\tau)| &= \left| u^*(\tau_{r-1}) + \frac{u^*(\tau_r) - u^*(\tau_{r-1})}{\tau_r - \tau_{r-1}} (\tau - \tau_{r-1}) - u^*(\tau) \right| \\ &= \left| \frac{\tau - \tau_{r-1}}{\tau_r - \tau_{r-1}} (u^*(\tau_r) - u^*(\tau)) + \frac{\tau_r - \tau}{\tau_r - \tau_{r-1}} (u^*(\tau_{r-1}) - u^*(\tau)) \right| \\ &\leq \left| \frac{\tau - \tau_{r-1}}{\tau_r - \tau_{r-1}} (u^*(\tau_r) - u^*(\tau)) \right| + \left| \frac{\tau_r - \tau}{\tau_r - \tau_{r-1}} (u^*(\tau_{r-1}) - u^*(\tau)) \right| \\ &\leq \frac{\tau - \tau_{r-1}}{\tau_r - \tau_{r-1}} |G(\tau_r - \tau_{r-1})| + \frac{\tau_r - \tau}{\tau_r - \tau_{r-1}} |G(\tau_r - \tau_{r-1})| \\ &= G(\tau_r - \tau_{r-1}) = G, \end{aligned}$$

which gives Equation (24). Since $u^*(0) = 0$ and $u^*(R) = 1$, we have $G \geq 1/R$. \square

PROOF OF THEOREM 3. By Corollary 2, the PLA of the ambiguity sets for utility functions does not impact the optimal value z^* , i.e., $z^* = z^{L*}$. Let $r^\# = R + 1 - r$. Then, by Theorem 2, for any $(i, j, r) \in \mathcal{H}$,

we have

$$\begin{aligned}
|w_{ijr}^{K^*} - w_{ijr}^*| &= \left| \frac{Rz^{K^*}U_{ijr}^{K^*}}{t_i \underline{s}_{ij}} - \frac{Rz^*U_{ijr}^*}{t_i \underline{s}_{ij}} \right| = \frac{Rz^{K^*}}{t_i \underline{s}_{ij}} |U_{ijr}^{K^*} - U_{ijr}^*| \\
&= \frac{Rz^{K^*}}{t_i \underline{s}_{ij}} \left| \frac{u_{ij}^{K^*}(r^\#)}{\sum_{r=1}^R u_{ij}^{K^*}(r)} - \frac{u_{ij}^*(r^\#)}{\sum_{r=1}^R u_{ij}^*(r)} \right| \\
&= \frac{Rz^{K^*}}{t_i \underline{s}_{ij}} \left| \frac{u_{ij}^{K^*}(r^\#) \left(\sum_{r=1}^R u_{ij}^*(r) \right) - u_{ij}^*(r^\#) \left(\sum_{r=1}^R u_{ij}^{K^*}(r) \right)}{\left(\sum_{r=1}^R u_{ij}^{K^*}(r) \right) \left(\sum_{r=1}^R u_{ij}^*(r) \right)} \right| \\
&= \frac{Rz^{K^*}}{t_i \underline{s}_{ij}} \left| \frac{u_{ij}^{K^*}(r^\#) \left(\sum_{r=1}^R u_{ij}^*(r) - \sum_{r=1}^R u_{ij}^{K^*}(r) \right)}{\left(\sum_{r=1}^R u_{ij}^{K^*}(r) \right) \left(\sum_{r=1}^R u_{ij}^*(r) \right)} + \frac{u_{ij}^{K^*}(r^\#) - u_{ij}^*(r^\#)}{\sum_{r=1}^R u_{ij}^*(r)} \right| \\
&\leq \frac{Rz^{K^*}}{t_i \underline{s}_{ij}} \left| \frac{u_{ij}^{K^*}(r^\#) \left(\sum_{r=1}^R u_{ij}^*(r) - \sum_{r=1}^R u_{ij}^{K^*}(r) \right)}{\left(\sum_{r=1}^R u_{ij}^{K^*}(r) \right) \left(\sum_{r=1}^R u_{ij}^*(r) \right)} \right| + \frac{Rz^{K^*}}{t_i \underline{s}_{ij}} \left| \frac{u_{ij}^{K^*}(r^\#) - u_{ij}^*(r^\#)}{\sum_{r=1}^R u_{ij}^*(r)} \right| \\
&\leq \frac{Rz^{K^*}u_{ij}^{K^*}(r^\#)}{t_i \underline{s}_{ij}} \left| \frac{1}{\sum_{r=1}^R u_{ij}^{K^*}(r)} - \frac{1}{\sum_{r=1}^R u_{ij}^*(r)} \right| + \frac{Rz^{K^*}G}{t_i \underline{s}_{ij} \sum_{r=1}^R u_{ij}^*(r)}.
\end{aligned}$$

Given that u is concave, $\sum_{r=1}^R u_{ij}^*(r) \in [(R-1)/2, R]$. When $\sum_{r=1}^R u_{ij}^{K^*}(r) \geq \sum_{r=1}^R u_{ij}^*(r)$, we have

$$\begin{aligned}
|w_{ijr}^{K^*} - w_{ijr}^*| &\leq \frac{Rz^{K^*}u_{ij}^{K^*}(r^\#)}{t_i \underline{s}_{ij}} \left(\frac{1}{\sum_{r=1}^R u_{ij}^*(r)} - \frac{1}{\sum_{r=1}^R u_{ij}^{K^*}(r)} \right) + \frac{Rz^{K^*}G}{t_i \underline{s}_{ij}} \left(\frac{1}{\sum_{r=1}^R u_{ij}^*(r)} \right) \\
&= \frac{Rz^{K^*} \left(G + u_{ij}^{K^*}(r^\#) \right)}{t_i \underline{s}_{ij}} \left(\frac{1}{\sum_{r=1}^R u_{ij}^*(r)} \right) - \frac{Rz^{K^*}u_{ij}^{K^*}(r^\#)}{t_i \underline{s}_{ij}} \left(\frac{1}{\sum_{r=1}^R u_{ij}^{K^*}(r)} \right),
\end{aligned}$$

which achieves maximum at $\sum_{r=1}^R u_{ij}^*(r) = (R-1)/2$.

When $\sum_{r=1}^R u_{ij}^{K^*}(r) \leq \sum_{r=1}^R u_{ij}^*(r)$, we have

$$|w_{ijr}^{K^*} - w_{ijr}^*| \leq \frac{Rz^{K^*} \left(G - u_{ij}^{K^*}(r^\#) \right)}{t_i \underline{s}_{ij}} \left(\frac{1}{\sum_{r=1}^R u_{ij}^*(r)} \right) - \frac{Rz^{K^*}u_{ij}^{K^*}(r^\#)}{t_i \underline{s}_{ij}} \left(\frac{1}{\sum_{r=1}^R u_{ij}^{K^*}(r)} \right).$$

which achieves optimal at $\sum_{r=1}^R u_{ij}^*(r) = R$ when $G \geq u_{ij}^{K^*}(r^\#)$, otherwise, $\sum_{r=1}^R u_{ij}^*(r) = (R-1)/2$. It follows that

$$\begin{aligned}
|w_{ijr}^{K^*} - w_{ijr}^*| &\leq \frac{z^{K^*}}{t_i \underline{s}_{ij}} \left(\max \left\{ \frac{2R \left(G + u_{ij}^{K^*}(r^\#) \right)}{R-1}, G - u_{ij}^{K^*}(r^\#) \right\} - Ru_{ij}^{K^*}(r^\#) \left(\frac{1}{\sum_{r=1}^R u_{ij}^{K^*}(r)} \right) \right) \\
&= \frac{Rz^{K^*}}{t_i \underline{s}_{ij}} \left(\frac{2 \left(G + u_{ij}^{K^*}(r^\#) \right)}{R-1} - \frac{u_{ij}^{K^*}(r^\#)}{\sum_{r=1}^R u_{ij}^{K^*}(r)} \right),
\end{aligned}$$

which gives the upper bound in Theorem 3, where the last equation follows from the fact that for $u_{ij}^{K^*}(r^\#) \geq 0$ and $G \geq 0$, $\frac{2R}{R-1} \left(G + u_{ij}^{K^*}(r^\#) \right) \geq G - u_{ij}^{K^*}(r^\#)$. \square

Appendix D. Proofs for Section 4

PROOF OF PROPOSITION 7. Let $\eta_i \geq 0$ for all $i \in \mathcal{I}$. Then, Equation (23) is equivalent to

$$\min_{\mathbf{w} \in \mathcal{W}, \boldsymbol{\eta} \geq 0} \left\{ \sum_{i=1}^I \eta_i : f_{ijr}(\alpha z^{K^*}) \geq g_{ijr}(w_{ijr}, \tilde{t}_i) - \eta_i \|\tilde{t}_i - t_i\|, \forall \tilde{t}_i \in \mathcal{T}_i, (i, j, r) \in \mathcal{H} \right\}.$$

For $\forall (i, j, r) \in \mathcal{H}$, we focus on

$$\alpha R U_{ijr}^{K^*} z^{K^*} \geq \tilde{t}_i \underline{s}_{ij} w_{ijr} - \eta_i \|\tilde{t}_i - t_i\|, \quad \forall \tilde{t}_i \in \mathcal{T}_i,$$

with \tilde{t}_i and t_i are singletons, which is equivalent to

$$\begin{aligned} \alpha R U_{ijr}^{K^*} z^{K^*} &\geq \sup_{\tilde{t}_i \in \mathcal{T}_i} \{ \underline{s}_{ij} w_{ijr} \tilde{t}_i - \eta_i \|\tilde{t}_i - t_i\| \} \\ &= \sup_{\tilde{t}_i \in \mathcal{T}_i} \left\{ \underline{s}_{ij} w_{ijr} \tilde{t}_i - \sup_{\|y_{ijr}\|_* \leq \eta_i} y_{ijr} (\tilde{t}_i - t_i) \right\} \\ &= \sup_{\tilde{t}_i \in \mathcal{T}_i} \inf_{\|y_{ijr}\|_* \leq \eta_i} \{ \underline{s}_{ij} w_{ijr} \tilde{t}_i - y_{ijr} (\tilde{t}_i - t_i) \} \\ &= \inf_{\|y_{ijr}\|_* \leq \eta_i} \sup_{\tilde{t}_i \in \mathcal{T}_i} \{ \underline{s}_{ij} w_{ijr} \tilde{t}_i - y_{ijr} (\tilde{t}_i - t_i) \}, \end{aligned}$$

where the last equality follows from the fact that the function is affine in both decision variables. Consider the inner maximization problem. Let $Z_{ijr} = \sup_{\tilde{t}_i \in \mathcal{T}_i} \{ \underline{s}_{ij} w_{ijr} \tilde{t}_i - y_{ijr} \tilde{t}_i \}$ for all $(i, j, r) \in \mathcal{H}$, where $\mathcal{T}_i = [t_i, \bar{t}_i]$. By strong duality, we have

$$\begin{aligned} Z_{ijr} &= \min_{\varepsilon_i^1, \varepsilon_i^2} \bar{t}_i \varepsilon_i^1 - t_i \varepsilon_i^2, \\ &\text{s.t. } \underline{s}_{ij} w_{ijr} - y_{ijr} + \varepsilon_i^2 - \varepsilon_i^1 = 0, \\ &\varepsilon_i^1 \geq 0, \varepsilon_i^2 \geq 0. \end{aligned}$$

By above reformulation, together with normalization constraints and non-negative constraints, we have

$$\begin{aligned} \min_{\mathbf{w}, \boldsymbol{\eta}, \varepsilon^1, \varepsilon^2} \quad & \sum_{i=1}^I \eta_i, \\ \text{s.t. } \quad & \alpha R U_{ijr}^{K^*} z^{K^*} \geq \bar{t}_i \varepsilon_i^1 - t_i \varepsilon_i^2 + t_i (\underline{s}_{ij} w_{ijr} + \varepsilon_i^2 - \varepsilon_i^1), \quad \forall (i, j, r) \in \mathcal{H}, \\ & \eta_i \geq \underline{s}_{ij} w_{ijr} + \varepsilon_i^2 - \varepsilon_i^1, \quad \forall (i, j, r) \in \mathcal{H}, \\ & \eta_i \geq -\underline{s}_{ij} w_{ijr} - \varepsilon_i^2 + \varepsilon_i^1, \quad \forall (i, j, r) \in \mathcal{H}, \\ & \sum_{i=1}^I \sum_{j=1}^J \sum_{r=1}^R w_{ijr} = 1, \\ & w_{ijr} \geq 0, \eta_i \geq 0, \varepsilon_i^1 \geq 0, \varepsilon_i^2 \geq 0, \quad \forall (i, j, r) \in \mathcal{H}. \end{aligned}$$

which gives the final formulation in Proposition 7. □

PROOF OF THEOREM 4. For simplify the following statement of proof, we define the set

$$\kappa \left(\mathbf{z}^\# - \mathbf{g}'(\mathbf{w}, \tilde{t}) \right) := \left\{ \eta \geq 0 : z_{jr}^\# - g'_{jr}(w_{jr}, \tilde{t}) \geq \eta \|\tilde{t} - t\|, \forall \tilde{t} \in \mathcal{T}, (j, r) \in \mathcal{J} \times \mathcal{K} \right\},$$

and let $\mathbf{v} = \mathbf{z}^\# - \mathbf{g}'(\mathbf{w}, \tilde{t})$. Thus, we have $\phi(\mathbf{v}) = \sup \kappa(\mathbf{v})$.

1. Monotonicity. If $\mathbf{v}_1 \succeq \mathbf{v}_2$, then for $\forall k \in \kappa(\mathbf{v}_2)$, we have $k \in \kappa(\mathbf{v}_1)$. It follows that $\kappa(\mathbf{v}_2) \subseteq \kappa(\mathbf{v}_1)$.

Taking the supremum yields $\phi(\mathbf{v}_1) \geq \phi(\mathbf{v}_2)$.

2. Positive homogeneity. The case where $\lambda = 0$ is trivial. For $\lambda > 0$, we have

$$\begin{aligned} \phi(\lambda \mathbf{v}) &= \sup \left\{ \eta \geq 0 : \lambda \mathbf{v} \succeq \eta \|\tilde{t} - t\|, \forall \tilde{t} \in \mathcal{T} \right\}, \\ &= \sup \left\{ \eta \geq 0 : \mathbf{v} \succeq \frac{1}{\lambda} \eta \|\tilde{t} - t\|, \forall \tilde{t} \in \mathcal{T} \right\}, \\ &= \sup \left\{ \lambda \eta' \geq 0 : \mathbf{v} \succeq \eta' \|\tilde{t} - t\|, \forall \tilde{t} \in \mathcal{T} \right\}, \\ &= \sup \lambda \left\{ \eta' \geq 0 : \mathbf{v} \succeq \eta' \|\tilde{t} - t\|, \forall \tilde{t} \in \mathcal{T} \right\}, \\ &= \lambda \phi(\mathbf{v}). \end{aligned}$$

3. Superadditivity. For all $k_1 \in \kappa(\mathbf{v}_1)$ and $k_2 \in \kappa(\mathbf{v}_2)$, we have

$$\mathbf{v}_1 + \mathbf{v}_2 \succeq (k_1 + k_2) \|\tilde{t} - t\|, \quad \forall \tilde{t} \in \mathcal{T}.$$

Thus, $k_1 + k_2 \in \kappa(\mathbf{v}_1 + \mathbf{v}_2)$ for all $k_1 \in \kappa(\mathbf{v}_1)$ and $k_2 \in \kappa(\mathbf{v}_2)$. The superadditivity $\phi(\mathbf{v}_1 + \mathbf{v}_2) \geq \phi(\mathbf{v}_1) + \phi(\mathbf{v}_2)$ follows by taking the supremum.

4. Pro-robustness. If $\mathbf{v} \succeq 0$, then for all $\tilde{t} \in \mathcal{T}$ and $\eta \leq 0$, we have $\mathbf{v} \succeq 0 \succeq \eta \|\tilde{t} - t\|$. Taking the supremum gives $\phi(\mathbf{v}) = 0$.

For the upper-semi-continuity of $\phi(\mathbf{v})$, we can show that its upper level set $\{\mathbf{v} : \phi(\mathbf{v}) \geq a\}$ is closed for any $a \leq 0$. Take any converging sequence of $v_{jr} \in \mathbf{v}$ for any $(j, r) \in \mathcal{J} \times \mathcal{R}$ such that $\{v_{jr}^k\} \rightarrow v_{jr}$. Let $\mathbf{v} = [v_{jr}]^T = [v_{11}, \dots, v_{1R}, v_{21}, \dots, v_{2R}, \dots, v_{J1}, \dots, v_{JR}]^T$. For any fixed $a \leq 0$, we need to show that $\phi(\mathbf{v}) = \phi([v_{jr}]^T) \geq a$ if $\phi([v_{jr}^k]^T) \geq a$ for all $k > 0$. Since \mathcal{L} is the space of bounded real-valued functions, we have $|v_{jr}| \leq M$ for some $M > 0$ and all $(j, r) \in \mathcal{J} \times \mathcal{R}$. Hence, $\lim_{k \rightarrow +\infty} \phi([v_{jr}^k]^T) = \phi([v_{jr}]^T) = \phi(\mathbf{v})$, confirming that the upper level set $\{\mathbf{v} : \phi(\mathbf{v}) \geq a\}$ is closed for any $a \leq 0$. \square